M.Sc
Part - II

Paper - III

NUMERICAL

MATHEMATICS

ANALYSIS

University of Gujrat

New Campus, Hafiz Hayat, Gujrat

Course Title :	**Numerical Analysis-I**
Course Code :	MATH-401
Instructor :	Dr. Jamshad Ahmad
Recommended Books :	1. Numerical Analysis, Richard L. Burden, J. Douglas Faires, (9th Edition), 2010, Brooks/Cole Publishing Company. 2. Numerical Methods, S.R.K. Iyenger R.K. Jain, New Age International Publishers. 3. Dr.Saeed Akhtar Bhatti, Numerical analysis with C++, Edition Lates,t Publisher Urdu Bazar lahore
Reference Books:	4. Applied Numerical Analysis, Curtis F. Gerald, Patrick O, Wheatley, (7th Edition), 2003, Addison Education. 5. Numerical Methods for Mathematics, Science and Engineering, John H. Mathews, (4th Edition), 2004, Prentice Hall International. 6. Numerical Methods, V. N. Vedamurthy, Ch. S. N. Iyenger, 2002, Vikas Publishing House PVT Ltd.
Email ID :	jamshad.ahmad@uog.edu.pk

Lectures	32 sessions of 90 minutes each
Attendance Policy	**A minimum of 70% attendance is required for a student to be eligible to take the final examination.** The students with less than 70% of the attendance in a course shall be given the grade SA (Short Attendance) in such a course and shall not be allowed to take its End Term Exams and will have to reappear in the course to get the required attendance to be eligible to sit in the exam when the course is offered the next time.
Grading	The course will be evaluated on the basis of the following percentage: • Mid Term **25%** • Sessional work **25%** o Presentation/Practical 5%

16 WEEK PLAN

1	Introduction to Numerical Analysis, Number System and errors; Round off Errors and Computer Arithmetic. Error estimation.
2	Floating point Arithmetic, Algorithm and Convergence Solution of non Linear Equations: Iterative Methods and Convergence.
3	Bisection Methods, Fixed point iterative Method, **QUIZ 1**
4	Regular Falsi, Secant and Newton´s Method, + **Assignment#1**
5	Gaussian Elimination methods, Gauss-Jordan Method, **QUIZ 2**
6	Matrix Inversion Methods, Factorization(Doolittle, and Crout) Method and its various Forms + **Assignment#2**
7	Factorization(Cholesky) Method and its various Forms,
8	Iterative Methods: Jacobi method, Gauss-Seidel method, SOR Methods.
9	**MID TERM,** Ill-Condition system and condition number
10	Eigen values and Eigen Vectors, Power and Rayleigh Quotient method, **Assignment#3**
11	Interpolating and Polynomial Approximation: Difference Operators, Interpolation with unequal intervals: Lagrange´s Interpolation Formula. **QUIZ 3.**
12	Newton´s Divided Difference Formula, Error in Polynomial Interpolation.
13	Interpolation with equal intervals: Gregory Newton Forward/Backward Interpolation Formula, **QUIZ 4**
14	Gauss´s Forward/Backward Interpolation Formula, Stirling´s Formula, Laplace Everett´s Formula, **Assignment#4**
15	Bessel´ s Formula, Aitken´s Interpolation, Hermite Interpolation.
16	Presentations

NUMERICAL ANALYSIS

* Introduction:-

Most of mathematical problems can be solved by analytical methods, they give exact or true solution of problem but Sometimes it is rather difficult to Solve the problem by analytic method.

For example:-

$$\int_0^1 \frac{e^{-x^2} x^2 \sin^2 x}{\sqrt{1+x^2}} \, dx$$

Many such examples can be cited for which solution through analytical method is not possible or very complex that is not suitable according to our requirment. In this situation where analytical method does not work we use Numerical Methods.

Numerical methods give approximate Solution of problem in steps or in series.

In numarical analysis we study to Solve problems by various methods which can calculate the approximate Solution of problem. As approximation of Solution is near to exact Solution then there is error in numarical problem.

SECTION (I)

- **Errors:-**

An error in numerical problem, is difference between the exact value of the problem and its computed value.

Let x be actual value and $\bar{x}$ be computed value then error is

Error $= E = x - \bar{x}$

- **Source of Error:-**

The error that affect the mathematical computations may be divided into 5 types

(i) **Input Error:-**

The input data can be the result of measurements with limited accuracy or real number which must be represented with fixed numbers of digit.

Example: 2.56782

We write it as 2.567

(ii) **Modelling Error:-**

The discrepancy between the model and physical system is source of error which can be termed as error of problem.

Like friction, gravity etc.

(iii) **Rounding Error:-**

The error which arises when we use fixed number of digits in calculation.

If the calculating device cannot be handled of to t-digits, number can not be used in subsequent calculation. Then the resulting value must be rounding off.

(iv) **Trancation Error:-**

The error arises when an infinite process is replaced by a finite one.

For example: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

We use only $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$ to solve problem.

(v) **Initial Error:-**

It is also called human error. These errors can be expected in program itself, by operator.

Most of these errors can be minimized by proper planning.

Types of Errors :-

(i) **Absolute Error :-**

If x is the actual value and $\bar{x}$ is approximate value then absolute error is defined by

$$E_a = A.E = |x - \bar{x}|$$
$$= |\Delta x|$$

where $\Delta x = x - \bar{x}$

Example: $x = 4.83$, $\bar{x} = 4.832$

$$E_a = |x - \bar{x}|$$
$$= 0.002$$

Note :-

Generally if a number is correct upto n-decimal places it has a rounding error

$$A.E \leq \frac{1}{2} \times 10^{-n}$$

(ii) **Relative Error :-**

Let x be true value of a quantity and $\bar{x}$ be computed value then relative error is denoted and defined as

$$E_r = \frac{|x - \bar{x}|}{x} \quad \text{s.t } x \neq 0$$
$$= \frac{|\Delta x|}{x}$$

As $x \cong \bar{x}$

Then $E_r = \frac{|\Delta x|}{\bar{x}}$

As the measure of accuracy, relative error is more precise than absolute error. The size of absolute error depends on the unit, where as the relative is dimensionless quantity.

Note :

A decimal number correct to n-significant digit has

$$RE = E_r \leq 5 \times 10^{-n}$$

(iii) **Percentage Error :-**

Relative error is expressed in percentage is called percentage error. It is denoted and defined as

$$PE = E_p = R.E \times 100$$
$$= \frac{|x - \bar{x}|}{x} \times 100$$

Example: $x = 4.83$, $\bar{x} = 4.832$

$$E_a = 0.002$$
$$R.E = \frac{|x - \bar{x}|}{x} = \frac{0.002}{4.83} = 0.0041$$

$$P.E = R.E \times 100$$
$$= 0.0041 \times 100$$
$$= 0.41\ \%$$

It is also called probable error.

• Effect of Error in Arithmatic Operations:-

(i) Error in Addition:-

Let $\bar{x}_1$ and $\bar{x}_2$ be approximate value of x_1 and x_2. Also E_1 be error in x_1 and E_2 be error in x_2. Then

$$E_2 = x_2 - \bar{x}_2 \qquad E_1 = x_1 - \bar{x}_1$$

Let E be error of '$x_1 + x_2$'

$$E = (x_1 + x_2) - (\bar{x}_1 + \bar{x}_2)$$
$$= x_1 + x_2 - \bar{x}_1 - \bar{x}_2$$
$$= E_1 + \cancel{\bar{x}_1} + E_2 + \cancel{\bar{x}_2} - \cancel{\bar{x}_1} - \cancel{\bar{x}_2}$$
$$\because x_1 = E_1 + \bar{x}_1$$

$$E = E_1 + E_2$$

Taking absolute on both sides

$$|E| = |E_1 + E_2|$$
$$\Rightarrow |E| \le |E_1| + |E_2|$$

which shows that absolute error of a sum of numbers is not greater then the sum of absolute errors of numbers. In general. For n-numbers

$$|E| < |E_1| + |E_2| + \dots + |E_n|$$

Hence the absolute error of the sum of n-numbers doesn't exceed the sum of absolute error of numbers.

(ii) Error in Subtraction:-

Let $\bar{x}_1$ and $\bar{x}_2$ be approximate value of x_1 and x_2. Also E_1 and E_2 be error in x_1 and x_2 resp. Let E be error in "$x_1 - x_2$"

$$E = (x_1 - x_2) - (\bar{x}_1 - \bar{x}_2)$$
$$= x_1 - x_2 - \bar{x}_1 + \bar{x}_2$$
$$= E_1 + \cancel{\bar{x}_1} - E_2 - \cancel{\bar{x}_2} - \cancel{\bar{x}_1} + \cancel{\bar{x}_2}$$
$$= E_1 - E_2 \qquad \because E_1 = x_1 - \bar{x}_1$$
$$= E_1 + (-E_2) \qquad \Rightarrow x_1 = E_1 + \bar{x}_1$$
$$|E| = |E_1 + (-E_2)|$$
$$\le |E_1| + |-E_2|$$
$$|E| \le |E_1| + |E_2|$$

Thus the absolute error of numbers is not greater then the sum of absolute error of the numbers.

(iii) Error in Product:-

Let $\bar{x}_1$ and $\bar{x}_2$ be approximate value of x_1 and x_2. Let E_1 and E_2 be errors in x_1 and x_2 respectively.

$$E_1 = x_1 - \bar{x}_1$$
$$E_2 = x_2 - \bar{x}_2$$

Let E be error in '$x_1 . x_2$'

$$E = x_1 x_2 - \bar{x}_1 \bar{x}_2$$
$$= (E_1 + \bar{x}_1)(E_2 + \bar{x}_2) - \bar{x}_1 \bar{x}_2$$
$$= E_1 E_2 + E_1 \bar{x}_2 + E_2 \bar{x}_1 + \cancel{\bar{x}_1 \bar{x}_2} - \cancel{\bar{x}_1 \bar{x}_2}$$

$$= E_1(x_2 - E_2) + E_2(x_1 - E_1) + E_1E_2$$

$$= E_1x_2 - E_1E_2 + E_2x_1 - \cancel{E_1E_2} + \cancel{E_1E_2}$$

$$E = E_1x_2 + E_2x_1 - E_1E_2$$

Dividing both sides by x_1x_2

$$\frac{E}{x_1x_2} = \frac{E_1x_2}{x_1x_2} + \frac{E_2x_1}{x_1x_2} - \frac{E_1E_2}{x_1x_2}$$

$$= \frac{E_1}{x_1} + \frac{E_2}{x_2} - \frac{E_1E_2}{x_1x_2}$$

As $\frac{E_1E_2}{x_1x_2}$ is very small quantity so neglecting

$$\Rightarrow \frac{E}{x_1x_2} \simeq \frac{E_1}{x_1} + \frac{E_2}{x_2}$$

Taking Absolute both sides

$$\left|\frac{E}{x_1x_2}\right| \simeq \left|\frac{E_1}{x_1} + \frac{E_2}{x_2}\right|$$

$$\leq \left|\frac{E_1}{x_1}\right| + \left|\frac{E_2}{x_2}\right|$$

Thus the relative error of product is approximatly the sum of relative error of factors.

(iv) Error in Division :-

Let $\bar{x}_1$ and $\bar{x}_2$ be approximate values of x_1 and x_2. Let E_1 and E_2 be errors of x_1 and x_2 resp. Let E be error in x_1/x_2

$$E = \frac{x_1}{x_2} - \frac{\bar{x}_1}{\bar{x}_2}$$

$$= \frac{x_1}{x_2} - \frac{(x_1 - E_1)}{(x_2 - E_2)} \qquad \because \bar{x}_1 = x_1 - E_1$$

$$= \frac{x_1}{x_2} - (x_1 - E_1)(x_2 - E_2)^{-1}$$

$$= \frac{x_1}{x_2} - x_1\left(1 - \frac{E_1}{x_1}\right)\left(1 - \frac{E_2}{x_2}\right)^{-1} x_2^{-1}$$

$$= \frac{x_1}{x_2} - \frac{x_1}{x_2}\left(1 - \frac{E_1}{x_1}\right)\left(1 - \frac{E_2}{x_2}\right)^{-1}$$

$\because$ By binomial expension

$$= \frac{x_1}{x_2} - \frac{x_1}{x_2}\left(1 - \frac{E_1}{x_1}\right)\left(1 + \frac{E_2}{x_2} + \dots\right)$$

$$= \frac{x_1}{x_2} - \frac{x_1}{x_2}\left(1 - \frac{E_1}{x_1} + \frac{E_2}{x_2} + \dots\right)$$

$$= \cancel{\frac{x_1}{x_2}} - \cancel{\frac{x_1}{x_2}} + \frac{E_1}{x_2} - \frac{E_2x_1}{x_2^2} + \dots$$

Neglecting heigher powers of E_1 & E_2

$$\frac{E}{x_1/x_2} = \frac{E_1}{x_2}\cdot\frac{x_2}{x_1} - \frac{E_2x_1}{x_2^2}\cdot\frac{x_2}{x_1}$$

$$\left|\frac{x_2E}{x_1}\right| = \left|\frac{E_1}{x_1} + \left(-\frac{E_2}{x_2}\right)\right|$$

$$\leq \left|\frac{E_2}{x_2}\right| + \left|\frac{E_1}{x_1}\right| \qquad \text{(proved)}$$

(v) Error in Function of n-Variables:-

Let $y = f(x_1, x_2, \dots, x_n)$ be a function of n-variable and Let $\delta x_1, \delta x_2, \dots, \delta x_n$ be error in $x_1, x_2, \dots, x_n$ and δy be error in y.

Taking increment on both sides

$$y + \delta y = f(x_1 + \delta x_1, x_2 + \delta x_2, \dots, x_n + \delta x_n)$$

Now,

Using Taylor's theorem

$$y + \delta y = f(x_1, x_2, \dots, x_n) + \left(\delta x_1 \frac{\partial f}{\partial x_1} + \delta x_2 \frac{\partial f}{\partial x_2} + \dots \delta x_n \frac{\partial f}{\partial x_n}\right) + \dots$$

Neglecting square terms of $\delta x_1, \delta x_2, \dots, \delta x_n$ we have

$$\cancel{y} + \delta y = \cancel{y} + \left(\delta x_1 \frac{\partial f}{\partial x_1} + \delta x_2 \frac{\partial f}{\partial x_2} + \dots + \delta x_n \frac{\partial f}{\partial x_n}\right)$$

$$\delta y = \delta x_1 \frac{\partial f}{\partial x_1} + \delta x_2 \frac{\partial f}{\partial x_2} + \dots + \delta x_n \frac{\partial f}{\partial x_n}$$

we get change in y w.r.t n-Variables.

For absolute error we taking absolute on both sides we get

$$|\delta y| = \left|\delta x_1 \frac{\partial f}{\partial x_1} + \delta x_2 \frac{\partial f}{\partial x_2} + \dots + \delta x_n \frac{\partial f}{\partial x_n}\right|$$

$$|\delta y| \leq \left|\frac{\partial f}{\partial x_1}\right| |\delta x_1| + \left|\frac{\partial f}{\partial x_2}\right| |\delta x_2| + \dots + \left|\frac{\partial f}{\partial x_n}\right| |\delta x_n|$$

(As required).

* Question:-

Find the error in product of n-numbers?

Solution:-

Let $y = x_1 x_2 x_3 \dots x_n \rightarrow$ (i)

and δy be error in y then

$$\delta y = \frac{\partial y}{\partial x_1}\delta x_1 + \frac{\partial y}{\partial x_2}\delta x_2 + \dots + \frac{\partial y}{\partial x_n}\delta x_n \rightarrow \text{(ii)}$$

Taking Log on both sides of (i) we get

$$\ln y = \ln x_1 + \ln x_2 + \dots + \ln x_n$$

Differentate w.r.t x_1

$$\frac{1}{y}\frac{\partial y}{\partial x_1} = \frac{1}{x_1}$$

$$\Rightarrow \frac{\partial y}{\partial x_1} = \frac{y}{x_1}$$

Similarly; $\frac{\partial y}{\partial x_2} = \frac{y}{x_2}$

.... $\frac{\partial y}{\partial x_n} = \frac{y}{x_n}$

Put in eq (ii)

$$\delta y = \frac{y}{x_1}\delta x_1 + \frac{y}{x_2}\delta x_2 + \dots + \frac{y}{x_n}\delta x_n$$

For relative error we divide by 'y'

$$\frac{\delta y}{y} = \frac{\delta x_1}{x_1} + \frac{\delta x_2}{x_2} + \dots + \frac{\delta x_n}{x_n}$$

$\therefore$ For absolute Error

$$\left|\frac{\delta y}{y}\right| \le \left|\frac{\delta x_1}{x_1}\right| + \left|\frac{\delta x_2}{x_2}\right| + \cdots\cdots + \left|\frac{\delta x_n}{x_n}\right|$$

$$E_r \le E_{1r} + E_{2r} + \cdots\cdots + E_{nr}$$

Thus relative error of product of n-number can not exceed the algebric sum of there relative error

* Question :-

Find relative error of y where

$$y = a_1 x_1^{m_1} x_2^{m_2} \cdots\cdots x_n^{m_n}$$

Solution :-

Let δy be error in y

$$\delta y = \frac{\partial f}{\partial x_1}\delta x_1 + \delta x_2 \frac{\partial f}{\partial x_2} + \cdots\cdots + \frac{\partial f}{\partial x_n}\delta x_n \rightarrow (*)$$

As $y = a x_1^{m_1} x_2^{m_2} \cdots\cdots x_n^{m_n}$

taking Log on both sides

$$\ln y = \ln a + m_1 \ln x_1 + m_2 \ln x_2 + \cdots + m_n \ln x_n$$

Taking partial derivative w.r.t x_1

$$\frac{1}{y}\frac{\partial y}{\partial x_1} = \frac{m_1}{x_1}$$

$$\frac{\partial y}{\partial x_1} = \frac{y m_1}{x_1}$$

Similarly, $\frac{\partial y}{\partial x_2} = \frac{y m_2}{x_2}$

$$\vdots \qquad \vdots$$

$$\frac{\partial y}{\partial x_n} = \frac{y m_n}{x_n}$$

$(*) \Rightarrow$

$$\delta y = \frac{y m_1}{x_1}\delta x_1 + \frac{y m_2}{x_2}\delta x_2 + \cdots\cdots + \frac{y m_n}{x_n}\delta x_n$$

$$\left|\frac{\delta y}{y}\right| = \left|\frac{m_1}{x_1}\delta x_1 + \frac{m_2}{x_2}\delta x_2 + \cdots\cdots + \frac{m_n}{x_n}\delta x_n\right|$$

$$\le \left|\frac{m_1}{x_1}\right||\delta x_1| + \left|\frac{m_2}{x_2}\right||\delta x_2| + \cdots + \left|\frac{m_n}{x_n}\right||\delta x_n|$$

$$E_r \le m_1 E_{1r} + m_2 E_{2r} + \cdots + m_n E_{nr}$$

Thus the relative error of y cannot be exceed algebric sum of there relative error.

* Question:- Find relative error of $y = x^m$

Solution:- Let δy be error in y

$$\Rightarrow \delta y = \frac{\delta y}{\delta x}\delta x \rightarrow (i)$$

As $y = x^m$

$$\ln y = m \ln x$$

$$\frac{1}{y}\frac{\partial y}{\partial x} = m\frac{1}{x}$$

$$\frac{\partial y}{\partial x} = m\frac{y}{x}$$

Put in (i) $\delta y = m\frac{y}{x}\delta x$

$$\frac{\delta y}{y} = \frac{m y}{y}\cdot\frac{\delta x}{x}$$

$$\Rightarrow \left|\frac{\delta y}{y}\right| = m\left|\frac{\delta x}{x}\right|$$

$$\Rightarrow E = m E_r \qquad \text{(Ans)}$$

DEFINITION:- (Significant Figures)

Significant figures (or digits) of an approximate number is any of the digit 1, 2, 3, ___, 9 in its decimal representation, or any zero except when it is used to fix the decimal point or to fill the place of unknown or discarded digit.

Example:

(i) Number 0.00475 the significant figures are 4, 7, 5. First three zeros are used merely to fix the position of decimal point and indicate the place values of other digits and are therefore not significant.

(ii) Number 8309, all digits, including zero, are significant.

(iii) In number 0.0006070, the first four zeros are not significant. All remaining digits, including the other two zeros, are significant.

ROUNDING OF RULE :-

To round off or simply round a number to n significant figures or digits, drop all digits to right of n^{th} place, or replace them by zeros (or if the zeros are needed as place holder. In this rounding-off rule, note following:

(a) If the first of discarded digits is less then 5, leave remaining digits unchanged

(b) If first discard digit is exactly 5 and there are non-zero digits among those discarded, add 1 to last retained digit.

(c) If first discarded digit exceeds 5, add 1 to the last retained digit.

(d) If first discarded digit, however is exactly 5 and all other discarded digits are zeros, the last retained digit is left unchanged or is increased by 1 according as it is even or odd (the even-digit rule).

The Solution Of Equations In One Variable

(Linear, Non Linear, Algebraic, Transcedental)

The expression of the form
$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n$
where $a_0 \neq 0$, a_i is constant and n is positive integer, is called a polynomial of degree n in variable x.

$f(x) = 0$ is called an algebraic equation

Example: $2x^3 - x + 10 = 0$ is algebraic equation.

An equations which contain some other function such as trigonometric, logarithmic, exponential are called transcedental equation.

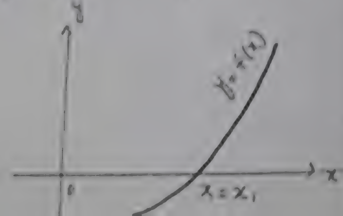
Example: $xe^x - \cos 2x = 0$

$x + \ln x = 0$.

The value of x which satisfies the equation $f(x) = 0$ is called root or Solution of equation.

Geometrically, a root of $f(x) = 0$ is that value of x where the graph of $y = f(x)$ Crosses x-axis

From fig $x = x_1$ is a root of equation $f(x) = 0$.



* **Intermediate Value Property :-** If $f(x)$ is continuous on $[a, b]$ and $f(a)$ and $f(b)$ have opposite sign then $f(x) = 0$ has atleast one real root between 'a' and 'b'.

* **METHODS FOR SOLVING EQUATIONS IN ONE VERIABLE**

We shall discuss the following methods for solving equations in one variable.

1. Bisection Method (OR) Interval halving Method.
2. Regula Falsi Method (OR) Method of False Position
3. Newton Raphson Method.
4. Iteration Method (OR) Fixed Point Iteration.
5. Secant Method.

(1) Bisection Method (OR) Interval Halving Method.

In order to solve $f(x)=0$ Consider $f(x)$ is continuous on $[a, b]$ and having opposite sign at $f(a)$ and $f(b)$.

i-e $f(a).f(b) < 0$

Then by intermediate value property there is a real root α of $f(x)=0$ in $]a, b[$

Let $c_1 = \frac{a+b}{2}$ be I^{st} approximation to the root.

If $f(c_1)=0$, then c_1 is required root.

If $f(c_1) \neq 0$ and

if $f(a)f(c_1) < 0$, then root lies in $]a, c_1[$

if $f(c_1)f(b) < 0$, then root lies in $]c_1, b[$.

Suppose root lies in $]c_1, b[$ then we find the mid point c_2 of $]c_1, b[$

i-e $c_2 = \frac{c_1+b}{2}$ and find $f(c_2)$.

If $f(c_2)=0$, then c_2 is required root.

If $f(c_2) \neq 0$ and

if $f(c_1)f(c_2) < 0$, then root lies in $]c_1, c_2[$

if $f(c_2)f(b) < 0$, then root lies in $]c_2, b[$

By having the intervals repeatedly we can find an approximation to the exact root of $f(x)=0$.

Example:-

Solve equation $x^3+x-1=0$ using Bisection method.

Solution:-

Let $f(x) = x^3+x-1$

x	0	0.2	0.4	0.6	0.8
$f(x)$	-1	-0.792	-0.536	-0.184	0.312

As sign change between 0.6 and 0.8

i-e $f(0.6) = -0.184 < 0$

$f(0.8) = 0.312 > 0$

As $f(0.6).f(0.8) < 0$, so root lies between $]0.6, 0.8[$.

I^{st} Iteration:

Let $a_1 = 0.6$, $b_1 = 0.8$, $c_1 = \frac{a+b}{2}$

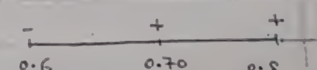
$\Rightarrow c_1 = \frac{0.6+0.8}{2} = 0.700$

$f(c_1) = f(0.700)$

$= (0.700)^3+(0.700)+1$

$= 0.0430$

$f(0.6)f(0.700) < 0$

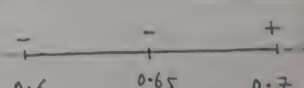


So root lies between $]0.6, 0.70[$.

2^{nd} Iteration:

$a_1 = 0.6$, $b_1 = 0.700$

$c_2 = \frac{0.6 + 0.700}{2} = 0.6500$



$f(c_2) = -0.0754$

$f(0.6500)f(0.700) < 0$

So root lies between $]0.65, 0.70[$.

3rd Iteration:

Let $a_3 = 0.6500$, $b_3 = 0.700$

$$c_3 = \frac{0.6500 + 0.700}{2} = 0.6750$$

$$f(c_3) = (0.6750)^3 + (0.6750) - 1$$
$$= -0.0175$$

0.65   0.675   0.70

As $f(0.6750)f(0.70) < 0$

so root lies between $]0.6750, 0.70[$

4th Iteration:

Let $a_4 = 0.6750$, $b_4 = 0.700$

$$c_4 = \frac{a_4 + b_4}{2} = \frac{0.6750 + 0.700}{2}$$

$\Rightarrow c_4 = 0.6875$

$$f(c_4) = (0.6875)^3 + (0.6874) - 1$$
$$= 0.0125$$

0.67   0.68   0.7

As $f(0.6750)f(0.6875) < 0$

so root lies between $]0.6750, 0.6875)$.

5th Iteration:

$a_5 = 0.6750$, $b_5 = 0.6875$

$$c_5 = \frac{0.6750 + 0.6875}{2}$$

$c_5 = 0.6813$

0.6750   0.6813   0.6875

$$f(c_5) = (0.6813)^3 + (0.6813) - 1$$
$$= -0.0025$$

As $f(0.6813)f(0.6875) < 0$

So root lies between $]0.6813, 0.6875[$

6th Iteration:

$a_6 = 0.6813$, $b_6 = 0.6875$

$$c_6 = \frac{0.6813 + 0.6875}{2} = 0.6844$$

$$f(c_6) = (0.6844)^3 + (0.6844) - 1$$
$$= 0.0050$$

0.6813   0.6844   0.6875

As $f(0.6813)f(0.6844) < 0$

so root lies between $]0.6813, 0.6844[$

7th Iteration:

$a_7 = 0.6813$, $b_7 = 0.6844$

$$c_7 = \frac{0.6813 + 0.6844}{2}$$

$c_7 = 0.6829$

0.6813   0.6829   0.6844

$$f(c_7) = (0.6829)^3 + (0.6829) - 1$$
$$= 0.0014$$

As $f(0.6813)f(0.6829) < 0$

So root lies between $]0.6813, 0.6829[$.

8th Iteration:

$a_8 = 0.6813$, $b_8 = 0.6829$

$$c_8 = \frac{0.6813 + 0.6829}{2}$$

$c_8 = 0.6821$

$f(x_8) = (0.6821)^3 + (0.6821) - 1$

$= -0.0007$

As $f(0.6821) f(0.6829) < 0$

So root lies between $]0.6821, 0.6829[$

__9th Iteration:__

$a_9 = 0.6821, \; b_9 = 0.6829$

$C_9 = \frac{0.6821 + 0.6829}{2}$

$C_9 = 0.6825$

$f(C_9) = (0.6825)^3 + (0.6825) - 1$

$= 0.0004$

So approximate root is 0.6825.

__Example:-__

Solve $f(x) = \sin x - 5x + 2$ by Bisection method.

__Solution:__ As $f(x) = \sin x - 5x + 2$

x	0	0.1	0.2	0.3	0.4	0.5	0.6
$f(x)$	2	1.5998	1.1986	0.7955	0.3894	-0.02	-0.435

As $f(x)$ changes sign between $(0.4, 0.5)$

i.e $f(0.4) = 0.3894 > 0$

$f(0.5) = -0.02 < 0$

$f(0.4) f(0.5) < 0$

So root lies between $]0.4, 0.5[$

__1st Iteration:__ $a_1 = 0.4, \; b_1 = 0.5$

$C_1 = \frac{0.4 + 0.5}{2} = 0.4500$

$f(C_1) = \sin(0.4500) - 5(0.4500) + 2$

$= 0.1850$

$f(0.45) f(0.5) < 0$

So root lies between $]0.4500, 0.5[$

__2nd Iteration:__

$a_2 = 0.4500, \; b_2 = 0.5$

$C_2 = \frac{0.4500 + 0.5}{2} = 0.4750$

$f(C_2) = \sin(0.4750) - 5(0.4750) + 2$

$= 0.0823$

As $f(0.4750) f(0.5) < 0$

So root lies between $]0.4750, 0.5[$

__3rd Iteration:__ $a_3 = 0.4750, \; b_3 = 0.5$

$C_3 = \frac{0.4750 + 0.5}{2} = 0.4875$

$f(C_3) = f(0.4875)$

$= \sin(0.4875) - 5(0.4875) + 2$

$= 0.0309$

$f(0.4875) f(0.5) < 0$

So root lies between $]0.4875, 0.5[$

__4th Iteration:__

$a_4 = 0.4875, \; b_4 = 0.5$

$C_4 = \frac{0.4875 + 0.5}{2} = 0.4938$

$f(c_4) = f(0.4938)$
$= \sin(0.4938) - 5(0.4938) + 2$
$= 0.0050$

As $f(0.4938) f(0.5) < 0$

So root lies between $]0.4938, 0.5[$

<u>5th Iteration:</u>

$a_5 = 0.4938$, $b_5 = 0.5$

$c_5 = \frac{0.4938 + 0.5}{2} = 0.4969$

$f(c_5) = f(0.4969)$
$= \sin(0.4969) - 5(0.4969) + 2$
$= -0.0078$

As $f(0.4969) f(0.4938) < 0$

So root lies between $]0.4938, 0.4969[$

<u>6th Iteration:</u>

$a_6 = 0.4938$, $b_6 = 0.4969$

$c_6 = \frac{0.4938 + 0.4969}{2} = 0.4954$

$f(c_6) = f(0.4954)$
$= \sin(0.4954) - 5(0.4954) + 2$
$= -0.0016$

As $f(0.4938) \cdot f(0.4954) < 0$

So root lies between $]0.4938, 0.4954[$

<u>7th Iteration:</u>

$a_7 = 0.4938$

$b_7 = 0.4954$

$c_7 = \frac{0.4938 + 0.4954}{2} = 0.4996$

$f(c_7) = f(0.4946)$
$= \sin(0.4946) - 5(0.4946) + 2$
$= 0.4946$

As $f(0.4946) \cdot f(0.4954) < 0$

So root lies between $]0.4946, 0.4954[$

<u>8th Iteration:</u>

$a_8 = 0.49\cancel{46}$

$b_8 = 0.4954$

$c_8 = \frac{0.4946 + 0.4954}{2} = 0.4950$

$f(c_8) = f(0.4950)$
$= \sin(0.4950) - 5(0.4950) + 2$
$= 0.0$

So approximate root is 0.4950.

* <u>Example:</u>

Use Bisection method to find the root of $2^x - 5x + 2 = 0$, Correct upto two decimal places.

<u>Solution:</u>

Let $f(x) = 2^x - 5x + 2$

x	0	1
$f(x)$	3	-1

As $f(0) \cdot f(1) < 0$

So root lies between $]0, 1[$

1st Iteration:

$a_1 = 0,\ b_1 = 1,\ c_1 = \frac{a_1+b_1}{2} = \frac{1}{2} = 0.5$

Now

$f(c_1) = f(0.5)$

$= (2)^{(0.5)} - 5(0.5) + 2$

$= 0.91$

As $f(0.5)f(1) < 0$ so root lies between $]0.5, 1[$.

2nd Iteration:

$a_2 = 0.5,\ b_2 = 1$

$c_2 = \frac{0.5+1}{2} = 0.75$

$f(c_2) = f(0.75)$

$= (2)^{(0.75)} - 5(0.75) + 2$

$= -0.07$

As $f(0.5).f(0.75) < 0$

so root lies between $]0.5, 0.75[$.

3rd Iteration:

$a_3 = 0.5,\ b_3 = 0.75$

$c_3 = \frac{0.5+0.75}{2} = 0.63$

$f(c_3) = f(0.63)$

$= (2)^{(0.63)} - 5(0.63) + 2$

$= 0.39$

As $f(0.63)f(0.75) < 0$, so root lies between $]0.63, 0.75[$

4th Iteration:

$a_4 = 0.63,\ b_4 = 0.75$

$c_4 = \frac{0.63+0.75}{2} = 0.70$

$f(c_4) = f(0.70)$

$= (2)^{0.70} - 5(0.70) + 2$

$= 0.12$

As $f(0.70)f(0.75) < 0$, so root lies between $]0.70, 0.75[$

5th Iteration:

$a_5 = 0.70,\ b_5 = 0.75$

$c_5 = \frac{0.70+0.75}{2} = 0.73$

$f(c_5) = f(0.73)$

$= (2)^{0.73} - 5(0.73) + 2$

$= 0.01$

As $f(0.73)f(0.75) < 0$, so root lies between $]0.73, 0.75[$

6th Iteration: $a_6 = 0.73,\ b_6 = 0.75$

$c_6 = \frac{0.73+0.75}{2} = 0.74$

$f(c_6) = f(0.74) = (2)^{0.74} - 5(0.74) + 2$

$= -0.03$

$\Rightarrow$ Correct upto two decimal places required root is 0.73.

Exercise

1- Use Bisection Method to Solve

(i) $2e^{-x} - \sin x = 0$

Ans: 0.9210

(2) $2x^3 + x - 2 = 0$

Ans: 0.4836

(3) $x + x^x = 100$

Ans: 3.5812

2- Given that $x = 1.45$ is exact solution of equation $x^2 - 5.57x + 5.974 = 0$. Apply 3 iterations of Bisection method for solving equation.

3- Find the root of equation $2x - \cos x - 3 = 0$ upto 3 decimal places by bisection method.

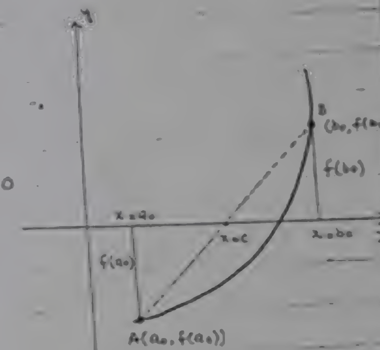
Ans: 1.5235.

2- Regula Falsi METHOD (OR) Method of False Position

If $f(a)f(b)<0$ and f is Continuous on $[a,b]$, then an approximation to the root of $f(x)=0$ on $[a,b]$ is given by

$$C = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

Proof:-



Let $f(x)$ be a Continuous function on $[a_0, b_0]$. Let $y = f(x) = 0$ to solve equation by using regula falsi method we find approximate root by draw Coard.

$f(a_0) < 0$, $f(b_0) > 0$

Let $A(a_0, f(a_0))$ and $B(b_0, f(b_0))$ be two points on a graph $y = f(x)$.

Draw a Coard AB.

Slope of line AB is $= \dfrac{f(b_0) - f(a_0)}{b_0 - a_0}$

and equation of line through A and B is given by

$$(y - f(b_0)) = \frac{f(b_0) - f(a_0)}{b_0 - a_0}(x - b_0)$$

As $x = c$ lies on x-axis at line AB $(c, 0)$ so it satisfies that equation

$$0 - f(b_0) = \frac{f(b_0) - f(a_0)}{b_0 - a_0}(c - b_0)$$

$$-f(b_0)(b_0 - a_0) = (f(b_0) - f(a_0))(c - b_0)$$

$$-\cancel{f(b_0)b_0} + f(b_0)a_0 = cf(b_0) - cf(a_0) - \cancel{b_0 f(b_0)} + b_0 f(a_0)$$

$$\Rightarrow a_0 f(b_0) - b_0 f(a_0) = c(f(b_0) - cf(a_0)$$

$$\Rightarrow a_0 f(b_0) - b_0 f(a_0) = c(f(b_0) - f(a_0))$$

$$\Rightarrow \frac{a_0 f(b_0) - b_0 f(a_0)}{f(b_0) - f(a_0)} = c$$

(i) If $f(c) = 0$, then we have exact root

(ii) If $f(c) \neq 0$

Then we consider the intervals

$[a_0, c)$ s.t $f(a_0)f(c) < 0$

(OR) $(c, b_0]$ s.t $f(c) \cdot f(b_0) < 0$.

We continue this process and makes new intervals in general for $[a_n, b_n]$ until we have exact root of equation.

$$\Rightarrow C = \frac{a_n f(b_n) - b_n f(a_n)}{f(b_n) - f(a_n)}$$

where $n = 0, 1, 2, \ldots\ldots$

1) Example :-

Solve $x\log_{10}x - 1.2 = 0$ correct upto 4 decimal places.

Solution :

Let $f(x) = x\log_{10}x - 1.2$

x	1	2	3
$f(x)$	-1.2	-0.5979	0.2314

As $f(2).f(3) < 0$ so root lis between 2 and 3.

Ist Iteration :

Let $a_1 = 2$, $b_1 = 3$

$f(a_1) = -0.5979$, $f(b_1) = 0.2314$

$$\Rightarrow c_1 = \frac{a_1 f(b_1) - b_1 f(a_1)}{f(b_1) - f(a_1)}$$

$$= \frac{2(0.2314) - 3(-0.5979)}{0.2314 + 0.5979}$$

$$= 2.7210$$

$$f(c_1) = f(2.7210)$$

$$= (2.7210)\log_{10}(2.7210) - 1.2$$

$$= -0.0171$$

As $f(2.7210)f(3) < 0$ so root lies between 2.7210 and 3.

2nd Iteration :

$a_2 = 2.7210$, $f(a_2) = -0.0171$

$b_2 = 3$, $f(b_2) = 0.2314$

$$\Rightarrow c_2 = \frac{a_2 f(b_2) - b_2 f(a_2)}{f(b_2) - f(a_2)}$$

$$= \frac{2.7210(0.2314) - 3(-0.0171)}{0.2314 + 0.0171}$$

$$= 2.7400$$

$$f(c_2) = f(2.7400)$$

$$= (2.7400)\log_{10}(2.7400) - 1.2$$

$$= -0.0006$$

As $f(2.7400) f(3) < 0$

so root lies between 2.7400 and 3.

3rd Iteration:

$$a_3 = 2.7400 \quad , \quad b_3 = 3$$

$$f(a_3) = -0.0006 \quad , \quad f(b_3) = 0.2314$$

$$c_3 = \frac{a_3 f(b_3) - b_3 f(a_3)}{f(b_3) - f(a_3)}$$

$$= \frac{2.7400(0.2314) - 3(-0.0006)}{0.2314 + 0.0006}$$

$$2.7405$$

$$f(c_3) = f(2.7405)$$

$$= (2.7405)\log_{10}(2.7405) - 1.2$$

$$= -0.0001$$

As $f(2.7405) f(3) < 0$

so root lies between 2.7405 and 3.

4th Iteration:

$$a_4 = 2.7405 \quad , \quad b_4 = 3$$

$$f(a_4) = -0.0001 \quad , \quad f(b_4) = 0.2314$$

$$c_4 = \frac{a_4 f(b_4) - b_4 f(a_4)}{f(b_4) - f(a_4)}$$

$$= \frac{2.7405(0.2314) - 3(-0.0001)}{0.2314 + 0.0001}$$

$$= 2.7408$$

$$f(c_4) = f(2.7408)$$

$$= (2.7408)\log_{10}(2.7408) - 1.2$$

$$= +0.0001$$

As $f(2.7405) f(2.7408) < 0$

so root lies between 2.7405 and 2.7408

5th Iteration:

$$a_5 = 2.7405 \quad , \quad b_5 = 2.7408$$

$$f(a_5) = -0.0001 \quad f(b_5) = 0.0001$$

$$\Rightarrow c_5 = \frac{a_5 f(b_5) - b_5 f(a_5)}{f(b_5) - f(a_5)}$$

$$= \frac{2.7405(0.0001) - 2.748(-0.0001)}{0.0001 + 0.0001}$$

$$= 2.7406$$

$$f(c_5) = f(2.7406)$$

$$= (2.7406)\log_{10}(2.7406) - 1.2$$

$$= 0.0000$$

$\Rightarrow 2.7406$ is the required root

Example:

Solve equation $x - e^{-x} = 0$ by regula falsi method ? Given root lies between $(0.5, 0.6)$.

Solution:-

Let $f(x) = x - e^{-x}$

As given $a_0 = 0.5$, $b_0 = 0.6$

$f(a_0) = -0.1065$; $f(b_0) = 0.0512$

I^{st} Iteration:

$$C_0 = \frac{a_0 f(b_0) - b_0 f(a_0)}{f(b_0) - f(a_0)}$$

$$= \frac{0.5(0.0512) - 0.6(-0.1065)}{0.0512 + 0.1065}$$

$C_0 = 0.5675$

$f(C_0) = (0.5675) - e^{-(0.5675)}$

$= 0.006$

$f(0.5) f(0.5675) < 0$

So root lies between $(0.5, 0.5675)$

2^{nd} Iteration:

$a = 0.5$, $b = 0.5675$

$f(a) = -0.1065$ $f(b) = 0.006$

$$C = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$= \frac{0.5(0.006) - 0.5675(-0.1065)}{0.006 + 0.1065}$$

$c = 0.5671$

$f(c) = f(0.5671)$

$= (0.5671) - e^{-(0.5671)}$

$= 0.0000$

So required root is 0.5671.

(Answer)

Example:-

Find Cube root of 5 using method of false position upto 3-decimal places.

Solution:- As $x = (5)^{1/3}$

$\Rightarrow x^3 = 5$

$\Rightarrow x^3 - 5 = 0$

$\Rightarrow f(x) = x^3 - 5$

x	1	1.2	1.4	1.6	1.8	2
$f(x)$	-4	-3.272	-2.256	-0.904	0.832	3

As $f(1.6) f(1.8) < 0$

So root lies between $(1.6, 1.8)$.

I^{st} Iteration:-

$a_1 = 1.6$, $b_1 = 1.8$

$f(a_1) = -0.904$ $f(b_1) = 0.832$

$$C_1 = \frac{a_1 f(b_1) - b_1 f(a_1)}{f(b_1) - f(a_1)}$$

$$= \frac{1.6(0.832) - 1.8(-0.904)}{0.832 + 0.904}$$

$= 1.741$

$f(c_1) = f(1.7041)$

$= (1.7041)^3 - 5$

$= -0.0514$

$\therefore f(1.8)\, f(1.7041) < 0$

$\therefore$ root lies between 1.8 and 1.7041

2nd Iteration:

$a_2 = 1.7041, \quad b_2 = 1.8$

$f(a_2) = -0.0514, \quad f(b_2) = 0.832$

$$c_2 = \frac{a_2 f(b_2) - b_2 f(a_2)}{f(b_2) - f(a_2)}$$

$$= \frac{1.7041(0.832) - (1.8)(-0.0514)}{0.832 + 0.0514}$$

$= 1.709$

$f(c_2) = f(1.709)$

$= (1.709)^3 - 5$

$= -0.009$

$f(1.709)\, f(1.8) < 0$

So root lies between 1.709 and 1.8

3rd Iteration:

$a_3 = 1.709$

$b_3 = 1.8$

$f(a_3) = -0.009$

$f(b_3) = 0.832$

$$\therefore c_3 = \frac{a_3 f(b_3) - b_3 f(a_3)}{f(b_3) - f(a_3)}$$

$$c_3 = \frac{1.709(0.832) - (1.8)(-0.009)}{0.832 + 0.009}$$

$= 1.710$

$f(c_3) = f(1.710)$

$= (1.710)^3 - 5$

$= 0.000$

required root is 1.710

(Answer)

EXERCISE

1- Solve by Regula falsi Method

(i) $3x - \cos x - 1 = 0$

Ans: 0.6071

(ii) $e^{-x} - \sin\left(\frac{\pi x}{2}\right) = 0$ $(\because \pi/2 = 90)$

Ans: 0.4436.

2- Apply Regula falsi method to find approximate root of $\cos x - x = 0$ correct upto 4 decimal places. on inteval $(0.5, \pi/4)$

Ans: 0.7391 $(\because \pi/4 = 0.7854)$

3. Given that $x = 1.45$ is exact root of equation $x^2 - 5.57x + 5.774 = 0$. Apply three iteration of Regula falsi method for Solving equation.

4- Approximate the solution to within 10^{-5} of following

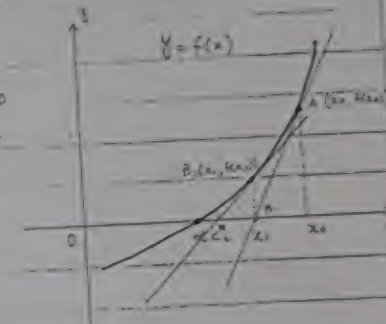
a) $3x^2 - e^x = 0$ Ans: 0.91001

b) $x^2 + 10\cos x = 0$ Ans: 1.96887

3- NEWTON-RAPHSON METHOD

Let α be exact root of equation $f(x)=0$ then the approximate root is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



Proof:- Let $f(x)=0$ be a function cuts the x-axis at α, so α be the exact root of equation. Let x_0 be a initial guess. The point on curve $A(x_0, f(x_0))$. Draw tangent at point A it cuts x-axis at point x_1 say $B(x_1, 0)$.

Slope of tangent A is

$\because$ slope $= m = \frac{dy}{dx}$, $m = \frac{y_2 - y_1}{x_2 - x_1}$

$$f'(x_0) = \frac{f(x_0) - 0}{x_0 - x_1}$$

$$x_0 - x_1 = \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Now point on curve is $(x_1, f(x_1))$.

Draw tangent at point B_1 cut the

axis at point C so slope of B.C is

$$f'(x_1) = \frac{0 - f(x_1)}{x_2 - x_1}$$

$$x_2 - x_1 = -\frac{f(x_1)}{f'(x_1)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Similarly;
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$\vdots \qquad \vdots$$

$$x_n = x_{n-1} - \frac{f(x_n)}{f'(x_n)}$$

Continue this process until we have root of problem.

Therefore

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$n = 0, 1, 2, \ldots$

Example:- Use Newton raphson method to find root of equation
$$\sin x = 1 - x.$$

Solution:

As $\sin x = 1 - x$

$\Rightarrow f(x) = \sin x + x - 1$

x	0	0.5	1
$f(x)$	-1	-0.02	0.8414

As $f(x) = \sin x + x - 1$

$f(x_n) = \sin x_n + x_n - 1$

$f'(x_n) = \cos x_n + 1$

Let initial guess root is 1 $(x_0 = 1)$

By Newton raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(\sin x_n + x_n - 1)}{\cos x_n + 1}$$

$$= \frac{x_n \cos x_n + \cancel{x_n} - \sin x_n - \cancel{x_n} + 1}{\cos x_n + 1}$$

$$x_{n+1} = \frac{x_n \cos x_n - \sin x_n + 1}{\cos x_n + 1} \rightarrow (*)$$

Ist Iteration:

When $n = 0$ in $(*)$

$$x_1 = \frac{x_0 \cos x_0 - \sin x_0 + 1}{\cos x_0 + 1}$$

$$x_1 = \frac{1\cos(1) - \sin(1) + 1}{\cos(1) + 1}$$
$$= 0.4537$$

2nd Iteration:

when $n = 1$

$$x_2 = \frac{x_1 \cos x_1 - \sin x_1 + 1}{\cos x_1 + 1}$$
$$= \frac{(0.4537)\cos(0.4537) - \sin(0.4537) + 1}{\cos(0.4537) + 1}$$
$$= 0.5106$$

3rd Iteration:

Put $n = 2$

$$x_3 = \frac{(0.5106)\cos(0.5106) - \sin(0.5106) + 1}{\cos(0.5106) + 1}$$
$$= 0.5110$$

4th Iteration:

Put $n = 3$

$$x_4 = \frac{x_3 \cos x_3 - \sin x_3 + 1}{\cos x_3 + 1}$$
$$= \frac{(0.5110)\cos(0.5110) - \sin(0.5110) + 1}{\cos(0.5110) + 1}$$
$$x_4 = 0.5110$$

As $x_3 = 0.5110$ and $x_4 = 0.5110$

In Newton-Raphson method repeated root is called root or Solution of equation.

$\Rightarrow$ Required root is 0.5110)

Example:-

Apply Newton Raphson Method to find solution of $x^3 + x - 1 = 0$.

Solution:- $f(x) = x^3 + x - 1$

x	-1	0	1
$f(x)$	-3	-1	1

Root lies between $(0, 1)$

Let Guess root $x_0 = 0$

As $f(x) = x^3 + x - 1$

$f'(x) = 3x^2 + 1$

$f'(x_n) = 3x_n^2 + 1$

By Newton Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$= x_n - \frac{(x_n^3 + x_n - 1)}{3x_n^2 + 1}$$
$$= \frac{3x_n^3 + \cancel{x_n} - x_n^3 - \cancel{x_n} + 1}{3x_n^2 + 1}$$
$$x_{n+1} = \frac{3x_n^3 - x_n^3 + 1}{3x_n^2 + 1}$$
$$\Rightarrow x_{n+1} = \frac{2x_n^3 + 1}{3x_n^2 + 1} \longrightarrow (i)$$

Ist Iteration:

Put $n = 0$ in (i)

$$x_1 = \frac{2x_0^3 + 1}{3x_0^2 + 1}$$
$$= \frac{1}{1} = 1$$

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<u>2nd Iteration:</u>

Put $n=1$ in (i)

$$x_2 = \frac{2x_1^3+1}{3x_1^2+1}$$

$$= \frac{2(1)+1}{3(1)+1} = \frac{3}{4} = 0.75$$

<u>3rd Iteration:</u>

Put $n=2$ in (i)

$$x_3 = \frac{2x_2^3+1}{3x_2^2+1}$$

$$= \frac{2(0.75)^3+1}{3(0.75)^2+1} = 0.6864$$

<u>4th Iteration:</u>

Put $n=3$ in (i)

$$x_4 = \frac{2x_3^3+1}{3x_3^2+1}$$

$$= \frac{2(0.6864)^3+1}{3(0.6864)^2+1} = 0.6823$$

<u>5th Iteration:</u>

Put $n=4$ in (i)

$$x_5 = \frac{2(0.6823)^3+1}{3(0.6823)^2+1}$$

$$x_5 = 0.6823$$

Hence the root is 0.6823.

(Answer).

<u>Example:-</u>

Using Newton's formula to find p^{th} root of N.

<u>Solution:-</u>

Let x be a p^{th} root of N

$\Rightarrow x = (N)^{1/p}$

$\Rightarrow x^p = N$

$\Rightarrow x^p - N = 0$

So $f(x) = x^p - N$

$f(x_n) = x_n^p - N$

$f'(x) = p\,x^{p-1}$

Now by Newton's formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(x_n^p - N)}{p\,x_n^{p-1}}$$

$$= \frac{p x_n^p - x_n^p + N}{p x_n^{p-1}}$$

$$= \frac{(p-1)x_n^p + N}{p\,x_n^{p-1}}$$

$$\Rightarrow x_{n+1} = \frac{1}{p}\left[(p-1)x_n + \frac{N}{x_n^{p-1}}\right]$$

(Answer).

Example:

Show that Newton's method of Square root of N is given by $x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$ and Also find

Square root of 11 by above formula upto 4 - decimal places.

Solution:-

Let

$$x = \sqrt{N}$$

$$x^2 = N$$

$$x^2 - N = 0$$

$$f(x) = x^2 - N$$

$$f(x_n) = x_n^2 - N$$

$$f'(x_n) = 2x_n$$

By Newton Raphson's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(x_n^2 - N)}{2x_n}$$

$$= \frac{2x_n^2 - x_n^2 + N}{2x_n}$$

$$= \frac{x_n^2 + N}{2x_n}$$

$$\Rightarrow x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right) \longrightarrow *$$

Also when $N = 11$

$$(*) \Rightarrow x_{n+1} = \frac{1}{2}\left(x_n + \frac{11}{x_n}\right) \longrightarrow (*')$$

Initial guess is $x_0 = 3$.

Put $n = 0$ in $*'$

$$x_1 = \frac{1}{2}\left(3 + \frac{11}{3}\right) = 3.3333$$

Put $n = 1$ in $*'$

$$x_2 = \frac{1}{2}\left(3.3333 + \frac{11}{3.3333}\right)$$

$$= 3.3167$$

Put $n = 2$ in $*'$

$$x_3 = \frac{1}{2}\left(3.3167 + \frac{11}{3.3167}\right)$$

$$= 3.3166$$

Put $n = 3$ in $*'$

$$x_4 = \frac{1}{2}\left(3.3166 + \frac{11}{3.3166}\right)$$

$$= 3.3166.$$

So approximate root is 3.3166

(Answer)

Exercise

1- By taking logarithm, find solution of $x^x = 10$ correct to 4 decimal places by Newton-Raphson Method

Ans: 2.5061

2- Using Newton-Raphson method, establish an iterative formula for finding a cube root of real number c.
Also find For $c = 30$ given that i.e $(30)^{1/3}$ initial guess is 3.

3- Find real root near to 1.5 of equation
$2x - 2\sin x - 1 = 0$

4- Find the negative root of equation
$e^x - \tan x = 0$ correct to 4 dp.

5- Using N-R method, evaluate a positive root of equation $e^x - 3x = 0$ correct to 3 decimal places.

Ans: 1.5121

6- Use Newton Raphson Method to find the reciprocal of a number N.
(Hint $x = \frac{1}{N} \Rightarrow xN - 1 = 0$)

4- # Iteration Method (OR) Fixed Point Iteration

Definition:-
The process of finding successive approximation to a quantity is called iteration.

* **Working Rule:-**
In order to solve $f(x) = 0$ by fixed point iteration method, first guess the interval $[a, b]$ s.t $f(x) = 0$ has a root within interval.
Next write the equation as
$x = g(x)$ s.t on $]a, b[$ for
$|g'(x)| < 1$ (convergence)

Then,
Setting the iteration as
$x_{n+1} = g(x_n)$
and by taking some initial guess x_0 and also by applying successive iterations the root can be obtained.

- Example:-

Find by fixed point iterative method to 5 dp root near 0.5 of equation $\sin x = 5x - 2$.

Solution:-

Let $f(x) = \sin x - 5x + 2$

x	0	0.5
$f(x)$	2	[illegible]

$\Rightarrow$ root lies between 0 and 0.5

Now,

$$\sin x = 5x - 2$$

$$\sin x + 2 = 5x$$

$$\Rightarrow x = \frac{1}{5}(\sin x + 2)$$

$$\Rightarrow g(x) = \frac{1}{5}(\sin x + 2)$$

$$g'(x) = \tfrac{1}{5}\cos x$$

$$|g'(x)| < 1 \quad \because |\cos x| < 1$$

so it is converges so setting iteration is

$$x_{n+1} = g(x_n)$$

$$\Rightarrow x_{n+1} = \frac{1}{5}(\sin x_n + 2) \rightarrow (i)$$

Let guess root is 0.4 be initial guess

1st Iteration:

Put $n = 0$ in (i)

$$\Rightarrow x_1 = \frac{1}{5}(2 + \sin x_0)$$

$$= \frac{1}{5}(2 + \sin(0.4))$$

$$= 0.4779$$

2nd Iteration:

Put $n = 1$ in (i)

$$x_2 = \frac{1}{5}(2 + \sin x_1)$$

$$= \frac{1}{5}(2 + \sin(0.4779))$$

$$= 0.4919$$

3rd Iteration:

Put $n = 2$ in (i)

$$x_3 = \frac{1}{5}(2 + \sin x_2)$$

$$= \frac{1}{5}(2 + \sin(0.4919))$$

$$= 0.4945$$

4th Iteration:

Put $n = 3$ in (i)

$$x_4 = \frac{1}{5}(2 + \sin x_3)$$

$$= \frac{1}{5}(2 + \sin(0.4945))$$

$$= 0.4949$$

5th Iteration:

Put $n = 4$ in (i)

$$x_5 = \frac{1}{5}(2 + \sin x_4)$$

$$= \frac{1}{5}(2 + \sin(0.4949))$$

$$= 0.4949$$

So $x = 0.4949$ is required root

(Answer).

- Example:- Find of fixed point iterative method to 5 dp root near 0.5 of equation $\sin x = 5x-2$.

Solution:-

Let $f(x) = \sin x - 5x + 2$

x	0	0.5
$f(x)$	2	

$\Rightarrow$ root lies between 0 and 0.5

Now

$$\sin x = 5x - 2$$

$$\sin x + 2 = 5x$$

$$\Rightarrow x = \frac{1}{5}(\sin x + 2)$$

$$\Rightarrow g(x) = \frac{1}{5}(\sin x + 2)$$

$$g'(x) = \tfrac{1}{5}\cos x$$

$$|g'(x)| < 1 \quad \because |\cos x| < 1$$

So it is converges so setting iteration is

$$x_{n+1} = g(x_n)$$

$$\Rightarrow x_{n+1} = \frac{1}{5}(\sin x_n + 2) \rightarrow (i)$$

Let guess root is 0.4 be initial guess

1st Iteration:

Put $n = 0$ in (i)

$$\Rightarrow x_1 = \frac{1}{5}(2 + \sin x_0)$$

$$= \frac{1}{5}(2 + \sin(0.4))$$

$$= 0.4779$$

2nd Iteration:

Put $n = 1$ in (i)

$$x_2 = \frac{1}{5}(2 + \sin x_1)$$

$$= \frac{1}{5}(2 + \sin(0.4779))$$

$$= 0.4919$$

3rd Iteration:

Put $n = 2$ in (i)

$$x_3 = \frac{1}{5}(2 + \sin x_2)$$

$$= \frac{1}{5}(2 + \sin(0.4919))$$

$$= 0.4945$$

4th Iteration:

Put $n = 3$ in (i)

$$x_4 = \frac{1}{5}(2 + \sin x_3)$$

$$= \frac{1}{5}(2 + \sin(0.4945))$$

$$= 0.4949$$

5th Iteration:

Put $n = 4$ in (i)

$$x_5 = \frac{1}{5}(2 + \sin x_4)$$

$$= \frac{1}{5}(2 + \sin(0.4949))$$

$$= 0.4949$$

So $x = 0.4949$ is required root.

(Answer).

Example:-

Find a real root correct to 4 dp of equation $x^3 + x^2 - 1 = 0$ by Iteration Method.

Solution:- Let $f(x) = x^3 + x^2 - 1$

x	0	1
$f(x)$	-1	1

As $f(0) = -1 < 0$

$f(1) = 1 > 0$

$\therefore$ root lies between 0 and 1.

Now $x^3 + x^2 - 1 = 0$

$$x^2(x+1) = 1$$

$$x^2 = \frac{1}{1+x}$$

$$x = \frac{1}{\sqrt{1+x}} \longrightarrow (i)$$

$$g(x) = (1+x)^{-1/2}$$

$$g'(x) = -\frac{1}{2}(1+x)^{-3/2}$$

$$= \frac{-1}{2(1+x)^{3/2}}$$

$$|g'(x)| = \left|\frac{-1}{2(1+x)^{3/2}}\right| < 1 \quad \text{for } x \in [0,1]$$

$\therefore$ Iteration method is applicable.

Setting iteration to (i)

$$x_{n+1} = \frac{1}{\sqrt{1+x_n}} \longrightarrow (*)$$

Now Guess root taken as

$$x_0 = 1.$$

1^{st} Iteration:

Put $n=0$ in (*)

$$x_1 = \frac{1}{\sqrt{1+x_0}} = \frac{1}{\sqrt{1+1}}$$

$$x_1 = 0.7071$$

2^{nd} Iteration:

Put $n=1$ in (*)

$$x_2 = \frac{1}{\sqrt{1+x_1}} = \frac{1}{\sqrt{1+0.7071}}$$

$$= 0.7654$$

3^{rd} Iteration:

Put $n=2$ in (*)

$$x_3 = \frac{1}{\sqrt{1+x_2}} = \frac{1}{\sqrt{1+0.7654}}$$

$$= 0.7526$$

4^{th} Iteration:

Put $n=3$ in (*)

$$x_4 = \frac{1}{\sqrt{1+x_3}}$$

$$= \frac{1}{\sqrt{1+0.7526}}$$

$$= 0.7554$$

5^{th} Iteration:

Put $n=4$ in (*)

$$x_5 = \frac{1}{\sqrt{1+x_4}}$$

$$= \frac{1}{\sqrt{0.7554+1}}$$

$$= 0.7548$$

6th Iteration:

Put $n=5$ in (*)

$$x_6 = \frac{1}{\sqrt{1+x_5}}$$

$$= \frac{1}{\sqrt{0.7548+1}}$$

$$= 0.7549$$

7th Iteration:

Put $n=6$ in (*)

$$x_7 = \frac{1}{\sqrt{1+x_6}}$$

$$= \frac{1}{\sqrt{0.7549+1}}$$

$$= 0.7549$$

Required root is 0.7549.

(Answer)

EXERCISE

1- Find the root of equation $2x - \cos x - 3 = 0$ upto 3 decimal places by simple iteration Method

Ans: $1.5235 \approx 1.524$

2- Solve $x + x^x = 100$ by iteration method upto 4 d.p.

Ans: 3.5812

3- Find the real root of equation $x - \sin x = 0.25$ using fixed point iteration

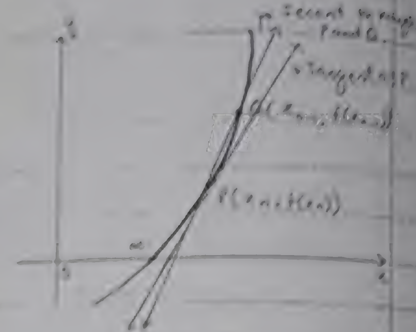
Ans: 1.1712

4- Set iterative formula for solving the equation $x^{6.1} - \cos x - 10 = 0$, as Starting with $x = 1.5$

Ans: 1.4611

1- SECANT METHOD:-

If x_{n-1} and x_n are the two approximation in the root α of $f(x)=0$, then in better approximation x_{n+1} to the root "α" is given by



$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Proof:-

Let $P(x_n, f(x_n))$ and $Q(x_{n-1}, f(x_{n-1}))$ be two point on the curve Then
Slope of tangent at P $\approx$ Slope of Secant through P and Q

$$\Rightarrow f'(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} \longrightarrow (i)$$

Also by Newton Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \longrightarrow (ii)$$

Put value $f'(x_n)$ in (ii)

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}}$$

$$\boxed{x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}}$$

which is required Secant formula.

Example:-

Find root of $x \log_{10} x - 1.2 = 0$ upto 4 decimal places by Secant Method.

Solution:-

Let $f(x) = x \log_{10} x - 1.2$

x	1	2	3
$f(x)$	-1.2	-0.5979	0.2314

$\Rightarrow$ root lies between 2 and 3.

Let $x_0 = 2$, $x_1 = 3$

$f(x_0) = -0.5979$, $f(x_1) = 0.2314$

Now by Secant method

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \longrightarrow (i)$$

Ist Iteration:

Put $n = 1$ in (i)

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{2(0.2314) - 3(-0.5979)}{0.2314 - (-0.5979)}$$

$x_2 = 2.7210$

$f(x_2) = f(2.7210)$

$= (2.7210)\log_{10}(2.7210) - 1.2$

$= -0.0171$

2nd Iteration:

Put $n=2$ in (i)

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{3(-0.0171) - (2.7210)(0.2314)}{-0.0171 - 0.2314}$$

$= 2.7404$

$f(x_3) = f(2.7404)$

$= (2.7404)\log_{10}(2.7404) - 1.2$

$= -0.0002$

3rd Iteration:

Put $n=3$ in (i)

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$$

$$= \frac{(2.7210)(-0.0002) - (2.7404)(-0.0171)}{-0.0002 - (-0.0171)}$$

$= 2.7406$

$f(x_4) = f(2.7406)$

$= (2.7406)\log_{10}(2.7406) - 1.2$

$= 0.0000$

So required root is 2.7406.

Example:-

Find positive root of equation $x^3 - x - 10 = 0$ by secant method.

Solution:- Let $f(x) = x^3 - x - 10$

x	0	0.5	1	1.5	2	2.5	3
$f(x)$	-10	-10.37	-10	-8.125	-4	3.125	14

$\Rightarrow$ root lies between 2 and 2.5.

$x_0 = 2$, $x_1 = 2.5$

$f(x_0) = -4$, $f(x_1) = 3.125$

Now by Secant Method

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n \cdot f(x_{n-1})}{f(x_n) - f(x_{n-1})} \rightarrow \text{(i)}$$

1st Iteration:

Put $n=1$ in (i)

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{2(3.125) - 2.5(-4)}{3.125 - (-4)}$$

$= 2.281$

$f(x_2) = f(2.281)$

$= (2.281)^3 - (2.281) - 10$

$= -0.413$

2nd Iteration:

Put $n=2$ in (i)

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{2.5(-0.413) - (2.281)(3.125)}{-0.413 - 3.125}$$

$x_3 = 2.307$

$f(x_3) = f(2.307)$

$= (2.307)^3 - (2.307) - 10$

$= -0.029$

3rd Iteration:

Put $n = 3$ in (i)

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$$

$$= \frac{2.281(-0.029) - (2.307)(-0.413)}{-0.029 - (-0.413)}$$

$= 2.309$

$f(x_4) = f(2.309)$

$= (2.309)^3 - (2.309) - 10$

$= (2.309)^3 - 12.309$

$= 0.001$

So approximation root is 2.309.

(Answer).

EXERCISE

1- By taking logarithm, find the solution of $x^x = 10$ correct to 4 significant figure by Secant method.

Ans : 2.5060

2- Using Secant method calculate a root of $x^3 + 2x^2 + 10x - 20 = 0$.

Ans: 1.3688

3- Solve equation $\sin x - 5x + 2 = 0$ by Secant method correct to 3 dp given root lies between 0.6 and 0.4:

Ans: 0.4950

4- Use Secant formula to compute a real root of equation $x^4 - x - 10 = 0$

5- Use secant Method find root of $\sin x = \frac{x}{2}$.

Ans: 0

6- Find smallest positive root of equation $f(x) = \cos x \cosh x - 1$ by secant method.

Ans: 4.730

• Order of Convergence:-

If e_n denotes the absolute error in n^{th} iteration and denoted the absolute error in $(n+1)^{th}$ iteration, then P is said to be order of Convergence if

$$e_{n+1} \propto e_n^P$$

i.e. $e_{n+1} = k e_n^P$ for some k.

• Order of Convergence of Fixed point iterative Method

Prove that order of Convergence of fixed point iteration method is one.

(OR)

Fixed point iteration method is linearly convergent.

Solution:

As given equation $f(x) = 0 \longrightarrow$ (i)

by iteration method we write it as

$$x = g(x) \longrightarrow \text{(ii)}$$

Setting iteration as

$$x_{n+1} = g(x_n) \longrightarrow \text{(iii)}$$

Let $x = a$ be root of equation (i)

$$\therefore f(a) = 0$$

Also from (ii)

$$a = g(a) \longrightarrow \text{(iv)}$$

Let e_n be error in n^{th} term and e_{n+1} be the error in $(n+1)^{th}$ term

$$x_n - a = e_n$$

$$x_{n+1} - a = e_{n+1}$$

$\Rightarrow x_n = e_n + a$, $x_{n+1} = e_{n+1} + a$

Put in equation (iii)

$$e_{n+1} + a = g(a + e_n)$$

By Taylor's Series

$$a + e_{n+1} = g(a) + e_n g'(a) + \frac{e_n^2}{2!} g''(a) + \cdots$$

$$= a + e_n g'(a)$$

$\because g(a) = a$

$\because$ Neglecting higher power of e_n.

$$a + e_{n+1} = a + e_n g'(a)$$

$$\Rightarrow e_{n+1} = e_n g'(a)$$

$$\Rightarrow e_{n+1} = g'(a) e_n^{(1)}$$

$$e_{n+1} = k e_n^{(1)} \quad \text{where } g'(a) = k$$

Which shows that order of convergence of fixed point iteration method is one. (OR)

Iteration method is linearly Convergent.

(proved)

Theorem:- Suppose that if equation $x = g(x)$ has a solution and continuous in $[a, b]$. If $|g'(x)| < k < 1$ for $x \in [a, b]$, then show that iteration $x_{i+1} = g(x_i)$ will converges for any starting point in $[a, b]$.

Proof:- Let $f(x) = 0$ be a given equation and we also write it as

$$x = g(x) \longrightarrow (i)$$

Setting the iteration

$$x_{n+1} = g(x_n) \longrightarrow (ii)$$

Let $x = a$ be the root of equation

i-e $f(a) = 0$

Also from (i)

$$a = g(a) \rightarrow (iii)$$

Let e_n and e_{n+1} be the error terms in n^{th} and $(n+1)^{th}$ iteration respectively

$$x_n - a = e_n$$

$$x_{n+1} - a = e_{n+1}$$

$$\Rightarrow x_n = a + e_n$$

and $x_{n+1} = a + e_{n+1}$

put in (ii) we get

$$a + e_{n+1} = g(a + e_n)$$

By Taylor's Series

$$a + e_{n+1} = g(a) + e_n g'(a) + \frac{e_n^2}{2!} g''(a) + \ldots\ldots$$

$$a + e_{n+1} = g(a) + e_n g'(a)$$

∵ By Neglecting Square and higher power of e_n.

$$\cancel{a} + e_{n+1} = \cancel{a} + e_n g'(a) \quad (\because a = g(a))$$

$$e_{n+1} = e_n g'(a)$$

$$|e_{n+1}| = |e_n||g'(a)|$$

$$\frac{|e_{n+1}|}{|e_n|} = g'(a).$$

If $|g'(a)| < 1$

$$\frac{|e_{n+1}|}{|e_n|} < 1$$

$$\Rightarrow |e_{n+1}| < |e_n|$$

The magnitude of error decreases in each step and the fixed point iteration method converges.

ORDER OF CONVERGENCE (Newton Raphson Method)

Theorem: Prove that order of Convergence of Newton-Raphson method is 2.

(OR)

Prove that Newton-Raphson method is quadratically Convergent.

Proof:- Let $x=a$ be root of the equation $f(x)=0$ and then $f(a)=0$.

Now; If we denote n^{th} approximation to root by x_n and error in n^{th} approx. to the root by e_n, then

$$e_n = x_n - a$$

and $e_{n+1} = x_{n+1} - a$

$$\Rightarrow x_n = e_n + a$$

$$x_{n+1} = e_{n+1} + a$$

Now by Newton-Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$e_{n+1} + \alpha = e_n + \alpha - \frac{f(a+e_n)}{f'(a+e_n)}$$

$$e_{n+1} = e_n - \frac{f(a) + e_n f'(a) + e_n^2/2!\, f''(a) + \cdots\cdots}{f'(a) + e_n f'(a) + \frac{e_n^2}{2!} f''(a) + \cdots\cdots}$$

$$e_{n+1} = e_n - \frac{e_n f'(a) + e_n^2/2!\, f''(a) + \cdots}{f'(a) + e_n f''(a) + \frac{e_n^2}{2!} f'''(a) + \cdots}$$

$\because f(a) = 0$

Neglecting square and higher power of e_n, we have:

$$e_{n+1} = e_n - \frac{e_n f'(a)}{f'(a) + e_n f''(a)}$$

$$= e_n - \frac{e_n f'(a)}{f'(a)\left[1 + e_n \frac{f''(a)}{f'(a)}\right]}$$

$$= e_n - e_n \left[1 + e_n \frac{f''(a)}{f'(a)}\right]^{-1}$$

$$= e_n - e_n \left[1 - e_n \frac{f''(a)}{f'(a)}\right]$$

$\because$ Neglecting remaining term

$$= \cancel{e_n} - \cancel{e_n} + e_n^2 \frac{f''(a)}{f'(a)}$$

$$e_{n+1} = e_n^2 \frac{f''(a)}{f'(a)}$$

$$e_{n+1} = k e_n^2 \quad , \quad k = \frac{f''(a)}{f'(a)}$$

$\Rightarrow$ Newton Raphson method has order of Convergence 2. i-e Newton Raphson method is quadratically Convergent

* Theorem:-

Prove that Newton-Raphson method is linearly Convergent if there exists multiple roots.

Proof:-

By Newton-Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \longrightarrow (i)$$

Let $x = a$ be root of $f(x) = 0$, then $f(a) = 0$. Now with usual meanings

$$e_n = x_n - a, \quad e_{n+1} = x_{n+1} - a$$

$$\Rightarrow x_n = e_n + a, \quad x_{n+1} = e_{n+1} + a$$

then (i) $\Rightarrow$

$$e_{n+1} + a = e_n + a - \frac{f(e_n + a)}{f'(e_n + a)}$$

$$e_{n+1} = e_n - \frac{f(a+e_n)}{f'(a+e_n)}$$

$$= e_n - \frac{f(a) + e_n f'(a) + e_n^2/2!\, f''(a) + \ldots\ldots}{f'(a) + e_n f''(a) + \frac{e_n^2}{2!} f'''(a) + \ldots\ldots}$$

Now if $x = a$ be a double root then. $f(a) = 0$ also $f'(a) = 0$ then

$$e_{n+1} = e_n - \frac{e_n^2/2!\, f''(a) + \ldots\ldots}{e_n f''(a) + \frac{e_n^2}{2!} f'''(a) + \ldots\ldots}$$

$$= e_n - \frac{e_n}{e_n}\left[\frac{e_n/2!\, f''(a) + \ldots\ldots}{f''(a) + \frac{e_n}{2!} f'''(a) + \ldots\ldots}\right]$$

$$= e_n - \frac{e_n}{2}\left[\frac{f''(a) + \ldots\ldots}{f''(a) + \frac{e_n}{2} f'''(a) + \ldots\ldots}\right]$$

$$= e_n - \frac{e_n}{2}\left[\frac{f''(a)}{f''(a)}\right] \quad \text{Neglecting remaining terms}$$

$$= e_n - \frac{e_n}{2}$$

$$e_{n+1} = \frac{1}{2} e_n^{(1)}$$

Continuing in this way if $x = a$ is a triple root then

$$e_{n+1} = \frac{1}{3} e_n^1$$

Also for multiplicity root n is given

$$e_{n+1} = \frac{1}{n} e_n$$

which shows that order of Convergence of Newton-Raphson for multiple roots is Linear.

(proved).

* Question:-

Show that Newton-Raphson method converges if $|f(x)| < \frac{|f'(x)|^2}{|f''(x)|}$

Solution:-

By Newton-Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \longrightarrow (i)$$

By Fixed point iteration method

$x_{n+1} = g(x_n) \rightarrow (ii)$

From (i) and (ii)

$$g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow g(x) = x - \frac{f(x)}{f'(x)}$$

$$\Rightarrow g'(x) = 1 - \frac{[f'(x)f'(x) - f(x)f''(x)]}{(f'(x))^2}$$

$$= \frac{\cancel{(f'(x))^2} - \cancel{(f'(x))^2} + f(x)f''(x)}{(f'(x))^2}$$

$$\boxed{|g'(x)|} = \left|\frac{f(x)f''(x)}{(f'(x))^2}\right|$$

For Convergence $|g'(x)| < 1$

$$\Rightarrow \frac{f(x)f''(x)}{(f'(x))^2} < 1$$

$$\Rightarrow |f(x)| < \frac{|f'(x)|^2}{|f''(x)|}$$

(proved)

* **Advantages of Newton-Rephson Method :-**

(i) It is fast convergent iterative method its convergence is quadratic.

(ii) Newton-Raphson method can be used to calculate the complex root of non-linear equation.

(iii) Newton's Method is useful in case of large value of $f'(x)$. i.e. when graph of function is $f(x)$ while crossing x-axis nearly vertical.

(iv) Newton's method is generally used to improve the result obtain by other methods.

(v) It is applicable to solution of both algebraic and transcedental function.

* **Disadvantages of Newton-Rephson Method:-**

(i) If two or more roots are equal or nearly equal, the method is not fastly convergent. In this case the convergence is Linear.

(ii) If root is very near to maximum or minimum value of function $f'(x)=0$ at such point Newton's method fails.

(iii) The evaluation of derivative $f'(x)$ at each step makes the method unpopular computationally and thus is not well adopted to computer usage.

Remark:-

The most important characteristics of order of convergence is that, that if an iteration method has large order of convergence then it will fastly converges to the root. From this we conclude that Newton-Raphson Method most rapidly converges to the root.

* ORDER OF CONVERGENCE (Secant Method).

Question:- Evaluate the rate of convergence of Secant Method (Error Analysis)?

Solution:- By Secant Method

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1}) f(x_n)}{f(x_n) - f(x_{n-1})} \rightarrow (i)$$

Let 'a' be the exact root of the equation $f(x)=0$, then $f(a)=0$.

Also let $x_{n-1} - a = e_{n-1}$

$x_n - a = e_n$

$\Rightarrow x_{n-1} = a + e_{n-1}$

$x_n = a + e_n$

and $x_{n+1} = a + e_{n+1}$

Using all these values in (i) we get.

$$a + e_{n+1} = a + e_n - \frac{[\cancel{a} + e_n - \cancel{a} - e_{n-1}] f(a+e_n)}{f(a+e_n) - f(a+e_{n-1})}$$

$$e_{n+1} = e_n - \frac{[e_n - e_{n-1}]\left[f(a) + e_n f'(a) + \frac{e_n^2}{2!} f''(a) + \dots\right]}{\left[f(a) + e_n f'(a) + \frac{e_n^2}{2!} f''(a) + \dots\right] - \left[f(a) + e_{n-1} f'(a) + \frac{e_{n-1}^2}{2!} f''(a) + \dots\right]}$$

As $f(a) = 0$ we get,

$$e_{n+1} = e_n - \frac{(e_n - e_{n-1})e_n f'(a)\left[1 + \frac{e_n}{2} \, f''(a)/f'(a) + \dots\right]}{(e_n - e_{n-1}) f'(a) + \left(\frac{e_n^2}{2} - \frac{e_{n-1}^2}{2}\right) f''(a) + \dots}$$

$$= e_n - \frac{e_n\left[1 + \frac{e_n}{2}\frac{f''(a)}{f'(a)} + \dots\right]}{\left[1 + \frac{e_n + e_{n-1}}{2}\frac{f''(a)}{f'(a)} + \dots\right]}$$

$$= e_n - e_n\left[1 + \frac{e_n}{2}\frac{f''(a)}{f'(a)} + \dots\right]\left[1 + \frac{e_n + e_{n-1}}{2}\frac{f''(a)}{f'(a)} + \dots\right]^{-1}$$

$$= e_n - e_n\left[1 + \frac{e_n}{2}\frac{f''(a)}{f'(a)}\right]\left[1 - \frac{e_n + e_{n-1}}{2}\frac{f''(a)}{f'(a)}\right]$$

∵ Neglecting remaining terms.

$$= e_n - e_n\left[1 + \frac{e_n}{2}\frac{f''(a)}{f'(a)} - \frac{e_n + e_{n-1}}{2}\cdot\frac{f''(a)}{f'(a)} - \frac{e_n}{2}\frac{f''(a)}{f'(a)}\cdot\frac{e_n + e_{n-1}}{2}\frac{f''(a)}{f'(a)}\right]$$

$$= \cancel{e_n} - \cancel{e_n} - \frac{e_n^2}{2}\frac{f''(a)}{f'(a)} + \frac{e_n(e_n + e_{n-1})}{2}\frac{f''(a)}{f'(a)} + \frac{e_n^2(e_n + e_{n-1})}{4}\left(\frac{f''(a)}{f'(a)}\right)^2$$

$$= -\frac{e_n^2}{2}\frac{\cancel{f''(a)}}{f'(a)} + \frac{e_n^2}{2}\frac{\cancel{f''(a)}}{f'(a)} + \frac{e_n e_{n-1}}{2}\frac{f''(a)}{f'(a)}$$

(∵ Neglecting remaining terms)

$$\Rightarrow e_{n+1} = \frac{1}{2}\frac{f''(a)}{f'(a)}\cdot e_n e_{n-1}$$

$$\Rightarrow e_{n+1} = K \cdot e_n e_{n-1}$$

where $K = \frac{1}{2}\frac{f''(a)}{f'(a)}$

$\Rightarrow e_{n+1} \propto e_n e_{n-1}$

This shows that in every iteration error is proportional to the product of the error in two previous iteration. Hence Secant method Converges.

* **Theorem:-**

Prove that order of Convergence of Secant Method is approximately 1.6180.

Proof:-

Let P be the order of Convergence of Secant Method, then with the usual meanings

$$e_{n+1} = K e_n^P \rightarrow \text{(i)}$$

Now also as secant method is (e_{n+1} proportional to $e_n e_{n-1}$)

i-e $e_{n+1} \propto e_n \cdot e_{n-1}$

So let $e_{n+1} = K_1 e_n e_{n-1} \rightarrow \text{(ii)}$

Now (i) can be written as

$$e_n = K e_{n-1}^P$$

$$\Rightarrow e_n^{1/P} = K^{1/P} e_{n-1}$$

$$\Rightarrow e_{n-1} = \frac{e_n^{1/P}}{K^{1/P}}$$

Put in (ii).

$$e_{n+1} = K_1 e_n \cdot \frac{e_n^{1/P}}{K^{1/P}}$$

$$\Rightarrow K e_n^P = \frac{K_1 e_n^{1+1/P}}{K^{1/P}} \quad (\because \text{By (i)}).$$

Comparing powers of e^n, we get

$$P = 1 + \frac{1}{P}$$

$$P^2 = 1 + P$$

$$\Rightarrow P^2 - P - 1 = 0$$

$$P = \frac{1 \pm \sqrt{1-4(1)(-1)}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

As P is positive

$$\rightarrow P = \frac{1+\sqrt{5}}{2}$$

$$P = 1.618$$

Hence order of convergence of secant method is 1.6180.

Remark:-

In Secant formula

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})f(x_n)}{f(x_n) - f(x_{n-1})}$$

If $f(x_n) = f(x_{n-1})$, Secant method fails. This shows that it does not converges always. This is a drawback of Secant method over the method of false position which always converges, But if Secant method converges its rate of convergence is faster then the method of false position.

* LU Decomposition (OR) Triangular Factorization Method (OR) Triangular Decomposition Method.

Given system of equation

$$A\underline{x} = \underline{b} \longrightarrow (i)$$

Decompose (or factorize) A into two square matrices L and U, where L is lower triangular matrix and U is upper triangular matrix, i-e $A = LU$

Equation (i) becomes

$$(LU)\underline{x} = \underline{b}$$

$$L(U\underline{x}) = \underline{b} \longrightarrow (ii)$$

Let $U\underline{x} = \underline{y}$

then (ii) becomes $L\underline{y} = \underline{b}$

Working Rule:-

(i) Express $A = LU$

(ii) From $L\underline{y} = \underline{b}$ Find $\underline{y}$.

(iii) From $U\underline{x} = \underline{y}$ Find $\underline{x}$.

In the decomposition $A = LU$, L and U are not unique. To find a unique decomposition, the diagonal entries of L or U are specified by

i) Dolittle's Method
ii) Crout's Method
iii) Cholesky's Method.

1) **DOLITTLE'S METHOD:-**

In Dolittle's method, all diagonal elements of L are restricted to be one.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

For inverse of matrix by Dolittle's method is

(i) $A = LU$

(ii) $A^{-1} = (LU)^{-1}$

$A^{-1} = U^{-1}L^{-1}$

$UA^{-1} = (UU^{-1})L^{-1}$

$UA^{-1} = L^{-1}$

(iii) $LL^{-1} = I$

$L^{-1} = ?$

Example:-

Solve the system of equation

$2x_1 + x_2 + 3x_3 = 11$

$4x_1 + 3x_2 + 10x_3 = 28$

$2x_1 + 4x_2 + 17x_3 = 31$

By Dolittle's method. Also find A^{-1}.

Solution:-

$$\underset{A}{\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 10 \\ 2 & 4 & 17 \end{bmatrix}} \underset{\underline{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{\underline{b}}{\begin{bmatrix} 11 \\ 28 \\ 31 \end{bmatrix}}$$

Let $LU = A$.

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 10 \\ 2 & 4 & 17 \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 10 \\ 2 & 4 & 17 \end{bmatrix}$$

Comparing Corresponding elements of matrix we get

$\Rightarrow \boxed{u_{11} = 2}$,

$\Rightarrow \boxed{u_{12} = 1}$,

$\Rightarrow \boxed{u_{13} = 3}$

$l_{21}u_{11} = 4$

$\Rightarrow \boxed{l_{21} = 2}$,

$l_{31}u_{11} = 2$,

$\Rightarrow \boxed{l_{31} = 1}$

$l_{21}u_{12} + u_{22} = 3$

$2(1) + u_{22} = 3$

$u_{22} = 3 - 2$

$\Rightarrow \boxed{u_{22} = 1}$

$l_{21}u_{13} + u_{23} = 10$

$(2)(3) + u_{23} = 10$

$\Rightarrow \boxed{u_{22} = 4}$

$l_{31}u_{12} + l_{32}u_{22} = 4$

$(1)(1) + l_{32}(1) = 4$

$l_{31} = 4 - 1$

$\Rightarrow \boxed{l_{31} = 3}$

$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 17$

$(1)(3) + (3)(4) + u_{33} = 17$

$u_{33} = 17 - 12 - 3$

$\Rightarrow \boxed{u_{33} = 2}$

we have

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

Let $Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 28 \\ 31 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ 2y_1 + y_2 \\ y_1 + 3y_2 + y_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 28 \\ 31 \end{bmatrix}$$

$\Rightarrow y_1 = 11$

$2y_1 + y_2 = 28$

$2(11) + y_2 = 28$

$\Rightarrow y_2 = 6$

and $y_1 + 3y_2 + y_3 = 31$

$(11) + 3(6) + y_3 = 31$

$\Rightarrow y_3 = 31 - 11 - 18$

$y_3 = 2$

$\therefore y = (11, 6, 2)^T$

Now, $Ux = y$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 6 \\ 2 \end{bmatrix}$$

$2x_1 + x_2 + 3x_3 = 11 \longrightarrow$ (i)

$x_2 + 4x_3 = 6 \longrightarrow$ (ii)

$2x_3 = 2 \longrightarrow$ (iii)

From above equation

$\Rightarrow x_3 = 1$

$\Rightarrow x_2 = 6 - 4(1) = 2$

$2x_1 + 2 + 3(1) = 11$

$2x_1 = 11 - 3 - 2$

$x_1 = \frac{6}{2} = 3$

$\Rightarrow x_1 = 3, \; x_2 = 2, \; x_3 = 1$

So solution vector is $x = (3, 2, 1)^T$

Also for A^{-1}:

First we find L^{-1}

$$\text{As } L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

$LL^{-1} = L^{-1}L = I$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ l_{21}' & 1 & 0 \\ l_{31}' & l_{32}' & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 + l_{21}' & 1 & 0 \\ 1 + 3l_{21}' + l_{31}' & 3 + l_{32}' & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Comparing corresponding elements we have

$2 + l_{21}' = 0, \quad 3 + l_{32}' = 0$

$\Rightarrow l_{21}' = -2 \quad \Rightarrow l_{32}' = -3$

and

$1 + 3l_{21}' + l_{31}' = 0$

$1 + 3(-2) + l_{31}' = 0$

$l_{31}' = 6 - 1$

$\Rightarrow l_{31}' = 5$

$\Rightarrow L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -3 & 1 \end{bmatrix}$

Now,

$UA^{-1} = L^{-1}$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2a_{11}+a_{21}+3a_{31} & 2a_{12}+a_{22}+3a_{32} & 2a_{13}+a_{23}+3a_{33} \\ a_{21}+4a_{31} & a_{22}+4a_{32} & a_{23}+4a_{33} \\ 2a_{31} & 2a_{32} & 2a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -3 & 1 \end{bmatrix}$$

$2a_{31} = 5$, $2a_{33} = 1$

$\Rightarrow a_{31} = 5/2$, $\Rightarrow a_{33} = 1/2$

$2a_{32} = -3$, $a_{21}+4a_{31} = -2$

$\Rightarrow a_{32} = -3/2$, $a_{21}+4(\frac{5}{2}) = -2$

$\Rightarrow a_{21} = -3\times4 = -12$

$a_{22}+4a_{32} = 1$, $a_{23}+4a_{33} = 0$

$a_{22} = 1-4(-3/2)$, $a_{23}+4(1/2) = 0$

$\Rightarrow a_{22} = 7$, $\Rightarrow a_{23} = -2.$

$2a_{11}+a_{21}+3a_{31} = 1$

$2a_{11}-3\times4+3(\frac{5}{2}) = 1$

$a_{11} = \frac{11}{4}$

$2a_{12}+a_{22}+3a_{32} = 0$, $2a_{13}+a_{23}+3a_{33} = 0$

$2(a_{12})+7+3(-3/2) = 0$, $2(a_{13})+(-2)+3(\frac{1}{2}) = 0$

$\Rightarrow a_{12} = -\frac{5}{4}$, $\Rightarrow a_{13} = 1/4$

$\Rightarrow A^{-1} = \begin{bmatrix} 11/4 & -5/4 & 1/4 \\ -12 & 7 & -2 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}.$

(Answer).

$$\Rightarrow L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -3 & 1 \end{bmatrix}$$

Now;

$$UA^{-1} = L^{-1}$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2a_{11}+a_{21}+3a_{31} & 2a_{12}+a_{22}+3a_{32} & 3a_{13}+a_{23}+3a_{33} \\ a_{21}+4a_{31} & a_{22}+4a_{32} & a_{23}+4a_{33} \\ 2a_{31} & 2a_{32} & 2a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -3 & 1 \end{bmatrix}$$

$2a_{31} = 5$, $2a_{33} = 1$

$\Rightarrow a_{31} = 5/2$, $\Rightarrow a_{33} = 1/2.$

$2a_{32} = -3$, $4a_{21} + 4a_{31} = -2$

$\Rightarrow a_{32} = -3/2$, $a_{21} + 4\left(\frac{5}{2}\right) = -2$

$\Rightarrow a_{21} = -3\times 4 = -12$

$a_{22} + 4a_{32} = 1$, $a_{23} + 4a_{33} = 0$

$a_{22} = 1 - 4(-3/2)$, $a_{23} + 4(1/2) = 0$

$\Rightarrow a_{22} = 7$, $\Rightarrow a_{23} = -2.$

$2a_{11} + a_{21} + 3a_{31} = 1$

$2a_{11} - 3^{\times 4} + 3\left(\frac{5}{2}\right) = 1$

$a_{11} = \frac{11}{4}$

$2a_{12} + a_{22} + 3a_{32} = 0$

$2(a_{12}) + 7 + 3(-3/2) = 0$

$\Rightarrow a_{12} = \frac{-1}{3}$

$3a_{13} + a_{23} + 3a_{33} = 0$

$3(a_{13}) + (-2) + 3\left(\frac{1}{2}\right) = 0$

$\therefore a_{13} = 1/4.$

$$\Rightarrow A^{-1} = \begin{bmatrix} 11/4 & -5/4 & 1/4 \\ -12 & 7 & -2 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

(Answer).

2) **Crout's Method :-**

In Crout's method, the upper triangular matrix U has unit diagonal elements.

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}, \quad U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

For inverse of matrix by Crout's Method :-

(i) $A = LU$

(ii) $A^{-1} = (LU)^{-1}$

$\Rightarrow \quad A^{-1} = U^{-1}L^{-1}$

$A^{-1}L = U^{-1}(L^{-1}L)$

$A^{-1}L = U^{-1}$

(iii) $UU^{-1} = U^{-1}U = I$

Find U^{-1}.

Example :-

Solve following system of equations

$$2x_1 + x_2 + 3x_3 = 11$$
$$4x_1 + 3x_2 + 10x_3 = 28$$
$$2x_1 + 4x_2 + 17x_3 = 31$$

by Crout's method Also find A^{-1} by using Crout's method.

Solution :-

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 10 \\ 2 & 4 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 28 \\ 31 \end{bmatrix}$$

$$A \quad \underline{x} = \underline{b}$$

Let $LU = A$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 10 \\ 2 & 4 & 17 \end{bmatrix}$$

$$\begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 10 \\ 2 & 4 & 17 \end{bmatrix}$$

Comparing corresponding elements of matrix.

$\Rightarrow l_{11} = 2$, $\quad l_{11}u_{12} = 1$

$\Rightarrow l_{21} = 4$, $\quad \Rightarrow u_{12} = 1/2$

$\Rightarrow l_{31} = 2$, $\quad l_{11}u_{13} = 3$

$\Rightarrow u_{13} = 3/2$

$l_{21}u_{12} + l_{22} = 3$, $\quad l_{21}u_{13} + l_{22}u_{23} = 10$

$(4)(\frac{1}{2}) + l_{22} = 3$, $\quad (4)(\frac{3}{2}) + 1\,u_{23} = 10$

$\Rightarrow l_{22} = 1$, $\quad \Rightarrow u_{23} = 4$.

$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 17$

$(2)(3/2) + 3(4) + l_{33} = 17$

$l_{33} = 17 - 12 - 3$

$\Rightarrow l_{33} = 2.$

So

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 3 & 2 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1/2 & 3/2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Let $Ly = b$

$$\begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 28 \\ 31 \end{bmatrix}$$

$$\begin{bmatrix} 2y_1 \\ 4y_1 + y_2 \\ 2y_1 + 3y_2 + 2y_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 28 \\ 31 \end{bmatrix}$$

$\Rightarrow y_1 = 11/2$

$4y_1 + y_2 = 28$

$\Rightarrow y_2 = 28 - 4\left(\frac{11}{2}\right) = 6$

$2y_1 + 3y_2 + 2y_3 = 31$

$2\left(\frac{11}{2}\right) + 3(6) + 2y_3 = 31$

$\Rightarrow y_3 = 1$

$y = [\,11/2,\ 6,\ 1\,]^T$

Now $Ux = y$

$$\begin{bmatrix} 1 & 1/2 & 3/2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11/2 \\ 6 \\ 1 \end{bmatrix}$$

$x_1 + \frac{1}{2}x_2 + \frac{3}{2}x_3 = \frac{11}{2} \longrightarrow (i)$

$x_2 + 4x_3 = 6 \longrightarrow (ii)$

$x_3 = 1 \longrightarrow (iii)$

Solving above equation

$\Rightarrow x_3 = 1$

$\Rightarrow x_2 = 6 - 4(1) = 2$

$\Rightarrow x_1 = \frac{11}{2} - 1 + x_3 \cdot \frac{3}{2}$

$= \frac{11}{2} - 1 - \frac{3}{2}$

$x_1 = 1 + 2 = 3$

So Solution Vector $x = [\,3,\ 2,\ 1\,]^T$

Now Also for A^{-1},

First we calculate U^{-1}.

As $UU^{-1} = U^{-1}U = I$

$$\begin{bmatrix} 1 & u_{12}' & u_{13}' \\ 0 & 1 & u_{23}' \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 3/4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} + u_{12}' & \frac{3}{4} + 4u_{12}' + u_{13}' \\ 0 & 1 & 4 + u_{23}' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow u_{12}' = -1/2$

$\Rightarrow u_{13}' = -\frac{3}{4} - 4\left(\frac{-1}{2}\right) = \frac{5}{4}$

$\Rightarrow u_{23}' = -4$

So

$$U^{-1} = \begin{bmatrix} 1 & -1/2 & 5/4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

Now

$A^{-1}L = U^{-1}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1/2 & 5/4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2a_{11} + 4a_{12} + 2a_{13} & a_{12} + 3a_{13} & 2a_{13} \\ 2a_{21} + 4a_{22} + 2a_{23} & a_{22} + 3a_{23} & 2a_{23} \\ 2a_{31} + 4a_{32} + 2a_{33} & a_{32} + 3a_{33} & 2a_{33} \end{bmatrix} = \begin{bmatrix} 1 & -1/2 & 5/4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow a_{13} = 1/4$, $a_{12} + 3a_{13} = -1/2 \rightarrow (iv)$

$\Rightarrow a_{23} = -2$, $a_{22} + 3a_{23} = 1 \rightarrow (v)$

$\Rightarrow a_{33} = 1/2$, $a_{32} + 3a_{33} = 0 \rightarrow (vi)$

$2a_{11} + 4a_{12} + 2a_{13} = 1 \longrightarrow (vii)$

$2a_{21} + 4a_{22} + 2a_{23} = 0 \longrightarrow (viii)$

$2a_{31} + 4a_{32} + 2a_{33} = 0 \longrightarrow (ix)$

From above equations we get

$a_{11} = 11/4$, $a_{12} = -5/4$

$a_{21} = -12$, $a_{22} = 7$

$a_{31} = 5/2$, $a_{32} = -3/2$

So $A^{-1} = \begin{bmatrix} 11/4 & -5/4 & 1/4 \\ -12 & 7 & -2 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$

(Answer).

3) CHOLESKY'S METHOD:-

In Cholesky's method, the diagonal elements of L and U are same, then for this method

$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$, $U = \begin{bmatrix} l_{11} & u_{12} & u_{13} \\ 0 & l_{22} & u_{23} \\ 0 & 0 & l_{33} \end{bmatrix}$

<u>Note:</u> If A is symmetric then:

$A = LL^T \quad \because U = L^T$

$A^{-1} = (L^{-1})^T L^{-1}$

<u>Example:-</u>

Solve by Cholesky's Method

$2x_1 + 1x_2 + 3x_3 = 11$

$4x_1 + 3x_2 + 10x_3 = 28$

$2x_1 + 4x_2 + 17x_3 = 31$.

<u>Solution:-</u>

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 10 \\ 2 & 4 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 28 \\ 31 \end{bmatrix}$$

$$A \quad \underline{x} = \underline{b}$$

Let $A = LU$

where

$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$ and $U = \begin{bmatrix} l_{11} & u_{12} & u_{13} \\ 0 & l_{22} & u_{23} \\ 0 & 0 & l_{33} \end{bmatrix}$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & u_{12} & u_{13} \\ 0 & l_{22} & u_{23} \\ 0 & 0 & l_{33} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 10 \\ 2 & 4 & 17 \end{bmatrix}$$

$$\begin{bmatrix} l_{11}^2 & l_{11}u_{12} & l_{11}u_{13} \\ l_{21}l_{11} & l_{21}u_{12}+l_{22}^2 & l_{21}u_{13}+l_{22}u_{23} \\ l_{31}l_{11} & l_{31}u_{12}+l_{32}l_{22} & l_{31}u_{13}+l_{32}u_{23}+l_{33}^2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 10 \\ 2 & 4 & 17 \end{bmatrix}$$

Comparing elements of matrices we get

$l_{11}^2 = 2$, $l_{11}u_{12} = 1$

$\Rightarrow l_{11} = \sqrt{2}$, $\Rightarrow u_{12} = 1/\sqrt{2}$

$l_{11}u_{13} = 3$, $l_{21}l_{11} = 4$

$\Rightarrow u_{13} = \frac{3}{\sqrt{2}}$, $\Rightarrow l_{21} = \frac{4}{\sqrt{2}}$

$l_{31}l_{11} = 2$, $\because l_{21}u_{12} + l_{22}^2 = 3$

$\Rightarrow l_{31} = \sqrt{2}$, $\frac{4}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + l_{22}^2 = 3$

$l_{21}u_{13} + l_{22}u_{23} = 10$, $\Rightarrow l_{22}^2 = 3-2$

$\frac{4}{\sqrt{2}} \cdot \frac{3}{\sqrt{2}} + 1u_{23} = 10$, $\Rightarrow l_{22} = 1$

$u_{23} = 10 - 6$

$\Rightarrow u_{23} = 4$

$l_{31}u_{12} + l_{32}l_{22} = 4$, $l_{31}u_{13} + l_{32}u_{23} + l_{33}^2 = 17$

$\sqrt{2} \cdot \frac{1}{\sqrt{2}} + (1)l_{32} = 4$, $\sqrt{2}\left(\frac{3}{\sqrt{2}}\right) + 3(4) + l_{33}^2 = 17$

$l_{32} = 4 - 1$

$\Rightarrow l_{32} = 3$, $l_{33}^2 = 17 - 12 - 3$

$l_{33}^2 = 2$

$\Rightarrow l_{33} = \sqrt{2}$

put in L and U we get

$$L = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 4/\sqrt{2} & 1 & 0 \\ \sqrt{2} & 3 & \sqrt{2} \end{bmatrix} , \quad U = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 3/\sqrt{2} \\ 0 & 1 & 4 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

Now $LY = b$

$$\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 4/\sqrt{2} & 1 & 0 \\ \sqrt{2} & 3 & \sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 28 \\ 31 \end{bmatrix}$$

$\sqrt{2}y_1 = 11 \Rightarrow y_1 = 11/\sqrt{2}$

$4/\sqrt{2}\, y_1 + y_2 = 28 \Rightarrow y_2 = 28 - 2$

$\sqrt{2}y_1 + 3y_2 + \sqrt{2}y_3 = 31$

$\Rightarrow \sqrt{2}\left(\frac{11}{\sqrt{2}}\right) + 3(6) + \sqrt{2}y_3 = 31$

$\Rightarrow \sqrt{2}y_3 = 31 - 18 - 11$

$= 2$

$\Rightarrow y_3 = \frac{2}{\sqrt{2}} = \sqrt{2}$

$Y = [11/\sqrt{2} \quad]^T$

$\therefore UX = Y$

$$\begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 3/\sqrt{2} \\ 0 & 1 & 4 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11/\sqrt{2} \\ 6 \\ \sqrt{2} \end{bmatrix}$$

$\sqrt{2}x_1 + \frac{1}{\sqrt{2}}x_2 + \frac{3}{\sqrt{2}}x_3 = \frac{11}{\sqrt{2}} \rightarrow$ (i)

$x_2 + 4x_3 = 6 \rightarrow$ (ii)

$\sqrt{2}x_3 = \sqrt{2}$

$\Rightarrow x_3 = 1$

From (2) $x_2 = 6 - 4(1) = 2$

From (1) $\sqrt{2}x_1 = \frac{11}{\sqrt{2}} - \frac{1}{\sqrt{2}}(2) - \frac{3}{\sqrt{2}}(1)$

$\Rightarrow x_1 = \frac{6}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}$

$\Rightarrow x_1 = \frac{6}{2} = 3$

$\therefore$ So Solution Vector is $\underline{x} = [3, 2, 1]^T$

(Answer).

Question:

Let $A = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix}$

(i) Find A^{-1} by Cholesky's method.

(ii) Hence solve system of equations

$x_1 + 2x_2 + 6x_3 = 13$

$2x_1 + 5x_2 + 15x_3 = 15$

$6x_1 + 15x_2 + 46x_3 = 19.$

Solution:-

As $A = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix}$

$A = A^T$

$\therefore$ A is symmetric, then

$A = LL^T$

Where $L^T = U$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix}$$

$$\begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix}$$

$l_{11}^2 = 1$, $\quad l_{21}l_{11} = 2$, $\quad l_{31}l_{11} = 6$

$\Rightarrow l_{11} = 1$ $\quad \Rightarrow l_{21} = 2$ $\quad \Rightarrow l_{31} = 6$

$l_{21}^2 + l_{22}^2 = 5$ $\qquad l_{21}l_{31} + l_{22}l_{32} = 15$

$(2)^2 + l_{22}^2 = 5$ $\qquad (2)(6) + (1)l_{32} = 15$

$l_{22}^2 = 5 - 4$ $\qquad l_{32} = 15 - 12$

$\Rightarrow l_{22} = 1$ $\qquad \Rightarrow l_{32} = 3$

$l_{31}^2 + l_{32}^2 + l_{33}^2 = 46$

$(6)^2 + (3)^2 + l_{33}^2 = 46$

$l_{33}^2 = 46 - 36 - 9$

$l_{33}^2 = 1$

$\Rightarrow l_{33} = 1.$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 3 & 1 \end{bmatrix}$, $\quad L^T = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

As we know that $LL^{-1} = L^{-1}L = I$

Next we calculate L^{-1}.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 3 & 1 \end{bmatrix} \begin{bmatrix} l_{11}' & l_{12}' & l_{13}' \\ l_{21}' & l_{22}' & l_{23}' \\ l_{31}' & l_{32}' & l_{33}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} l_{11}' & l_{12}' & l_{13}' \\ 2l_{11}'+l_{21}' & 2l_{12}'+l_{22}' & 2l_{13}'+l_{23}' \\ 6l_{11}'+3l_{21}'+l_{31}' & 6l_{12}'+3l_{22}'+l_{32} & 6l_{13}'+3l_{23}'+l_{33}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Comparing each element corresponding to matrix we get.

$\Rightarrow l_{11}' = 1$, $l_{12}' = 0$, $l_{13}' = 0$

$2l_{11}' + l_{21}' = 0$, $2l_{12}' + l_{22}' = 1$

$\Rightarrow l_{21}' = -2$ $\quad \Rightarrow l_{22}' = 1$

$2l_{13}' + l_{23}' = 0$, $6l_{11}' + 3l_{21}' + l_{31}' = 0$

$l_{23}' = 0$ $\quad 6(1) + 3(-2) + l_{31}' = 0$

$\Rightarrow l_{31}' = 0$

$6l_{12}' + 3l_{22}' + l_{32} = 0$

$6(0) + 3(1) + l_{32} = 0$

$\Rightarrow l_{32} = -3$

$6l_{13}' + 3l_{23}' + l_{33}' = 1$

$6(0) + 3(0) + l_{33}' = 1$

$\Rightarrow l_{33}' = 1$

$$\therefore L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$A = LL^T$

$A^{-1} = (LL^T)^{-1}$

$= (L^T)^{-1} L^{-1}$

$= (L^{-1})^T L^{-1}$

$\Rightarrow A^{-1} = (L^{-1})^T L^{-1}$

$$A^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4 & -2 & 0 \\ -2 & 1+9 & -3 \\ 0 & -3 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 10 & -3 \\ 0 & -3 & 1 \end{bmatrix}$$

(Answer).

(ii) $x_1 + 2x_2 + 6x_3 = 13$

$2x_1 + 5x_2 + 15x_3 = 15$

$6x_1 + 15x_2 + 46x_3 = 19$

$A\underline{x} = \underline{b}$

$\underline{x} = A^{-1}\underline{b}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 10 & -3 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 13 \\ 15 \\ 19 \end{bmatrix}$$

$$= \begin{bmatrix} 65 - 30 + 0 \\ -26 + 150 - 57 \\ 0 - 45 + 17 \end{bmatrix} = \begin{bmatrix} 35 \\ 67 \\ -26 \end{bmatrix}$$

(Answer).

$\underline{x} = [35, 67, -26]^T$.

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$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 3 & 1 \end{bmatrix} \begin{bmatrix} l_{11}' & l_{12}' & l_{13}' \\ l_{21}' & l_{22}' & l_{23}' \\ l_{31}' & l_{32}' & l_{33}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} l_{11}' & l_{12}' & l_{13}' \\ 2l_{11}'+l_{21}' & 2l_{12}'+l_{22}' & 2l_{13}'+l_{23}' \\ 6l_{11}'+3l_{21}'+l_{31}', & 6l_{12}'+3l_{22}'+l_{32} & 6l_{13}'+3l_{23}'+l_{33}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Comparing each element corresponding to matrix we get,

$\Rightarrow l_{11}' = 1$, $l_{12}' = 0$, $l_{13}' = 0$

$2l_{11}' + l_{21}' = 0$, $2l_{12}' + l_{22}' = 1$

$\Rightarrow l_{21}' = -2$ $\quad \Rightarrow l_{22}' = 1$

$2l_{13}' + l_{23}' = 0$, $6l_{11}' + 3l_{21}' + l_{31}' = 0$

$l_{23}' = 0$ $\quad 6(1) + 3(-2) + l_{31}' = 0$

$\Rightarrow l_{31}' = 0$

$6l_{12}' + 3l_{22}' + l_{32} = 0$

$6(0) + 3(1) + l_{32} = 0$

$\Rightarrow l_{32} = -3$

$6l_{13}' + 3l_{23}' + l_{33}' = 1$

$6(0) + 3(0) + l_{33}' = 1$

$\Rightarrow l_{33}' = 1.$

$$\therefore L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$A = LL^T$

$A^{-1} = (LL^T)^{-1}$

$= (L^T)^{-1} L^{-1}$

$= (L^{-1})^T L^{-1}$

$\Rightarrow A^{-1} = (L^{-1})^T L^{-1}$

$$A^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4 & -2 & 0 \\ -2 & 1+9 & -3 \\ 0 & -3 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 10 & -3 \\ 0 & -3 & 1 \end{bmatrix}$$ (Answer).

(ii) $x_1 + 2x_2 + 6x_3 = 13$

$2x_1 + 5x_2 + 15x_3 = 15$

$6x_1 + 15x_2 + 46x_3 = 19$

$A\underline{x} = b$

$\underline{x} = A^{-1}\underline{b}$

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$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 10 & -3 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 13 \\ 15 \\ 19 \end{bmatrix}$$

$$= \begin{bmatrix} 65-30+0 \\ -26+150-57 \\ 0-45+19 \end{bmatrix} = \begin{bmatrix} 35 \\ 67 \\ -26 \end{bmatrix}$$

(Answer).

$\underline{x} = [35, 67, -26]^T.$

Exercise

1): Factorization method apply on matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

in a form $A = LU$, where L is lower triangular and U is upper triangular matrices.

2): Solve the following system of equation by LU Decomposition by Crout's method.

$$3x_1 + 2x_2 + x_3 = 10$$
$$x_1 + 4x_2 - x_3 = 6$$
$$x_1 + 2x_2 + 5x_3 = 20$$

3): Find inverse of the following matrix

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -1 & 9 \\ -3 & 4 & -1 \end{bmatrix}$$

by expressing A in a form of LU-Decomposition by Dolittle's Method.

4): Solve the system of equations by factorizing the coefficient matrix approximatly

$$x_1 + 2x_2 + (-1)x_3 = 5$$
$$2x_1 + 3x_2 - 5x_3 = 13$$
$$3x_1 + 8x_2 + 6x_3 = 13$$

5): Write the matrix $A = \begin{bmatrix} 1.25 & 0.8 & 1.1 \\ 4.5 & 1.1 & 0.7 \\ 0.4 & 1.4 & 0.6 \end{bmatrix}$

as a product of an upper and a lower triangular matrices and hence invert it

6): Solve by Cholesky's method of system of equations

$$3x_1 + 2x_2 + x_3 = 10$$
$$x_1 + 4x_2 + x_3 = 6$$
$$2x_1 + x_2 + 4x_3 = 12.$$

7): Solve the following system of linear equation

$$2x_1 + 0x_2 + x_3 = 4$$
$$-3x_1 + 4x_2 - 2x_3 = -3$$
$$x_1 + 7x_2 - 5x_3 = 6$$

by

(i) Dolittle's Method

(ii) Crout's Method

(iii) Cholesky's Method.

LDU FACTORIZATION :-

LDU-Factorization is an other method to factorize any (square) matrix. To express $A = LDU$
where L and U are lower and upper triangular matrices with unit entries in diagonal and D is a diagonal matrix.

To find inverse by expressing $A = LDU$ is :

$$A = LDU$$
$$A^{-1} = (LDU)^{-1}$$
$$= U^{-1}D^{-1}L^{-1}$$
$$UA^{-1} = D^{-1}L^{-1}$$

where L^{-1} is given by $LL^{-1} = I$.
then find L^{-1} we have A^{-1}.

Example :
Obtain LDU factorization of matrix

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 2 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

Solution:-

$$LDU = A$$

where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$, $U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

and $D = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 2 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} d_{11} & d_{11}u_{12} & d_{11}u_{13} \\ d_{11}l_{21} & d_{11}l_{21}u_{12}+d_{22} & d_{11}l_{21}u_{13}+d_{22}u_{23} \\ d_{11}l_{31} & d_{11}l_{31}u_{12}+d_{22}l_{32} & d_{11}l_{31}u_{13}+d_{22}l_{32}u_{23}+d_{33} \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 2 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

By comparing we have :

$\Rightarrow d_{11} = 1$, $d_{11}u_{13} = -2$
$\Rightarrow u_{13} = -2.$

$\Rightarrow d_{11}u_{12} = -1$
$\Rightarrow u_{12} = -1$

$d_{11}l_{21} = 1$, $d_{11}l_{31} = 1$
$\Rightarrow l_{21} = 1$, $\Rightarrow l_{31} = 1$

$d_{11}l_{21}u_{12} + d_{22} = 2$, $d_{11}l_{21}u_{13} + d_{22}u_{23} = 1$
$1(1)(-1) + d_{22} = 2$, $(1)(1)(-2) + (3)u_{23} = 1$
$\Rightarrow d_{22} = 3$, $\Rightarrow u_{23} = 1$

$d_{11}l_{31}u_{12} + d_{22}l_{32} = 3$
$(1)(1)(-1) + 3\,l_{32} = 3$
$\Rightarrow l_{32} = 4/3$

$d_{11}l_{31}u_{13} + d_{22}l_{32}u_{23} + d_{33} = -1$
$(1)(1)(-2) + 3\left(\frac{4}{3}\right)(1) + d_{33} = -1$
$\Rightarrow d_{33} = -3.$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 4/3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

$$L \quad\quad D \quad\quad U \quad = \quad A$$

<u>Example:</u>

Find inverse of matrix A (define in previous example) by expressing in form of LDU.

<u>Solution:</u>

(*) Working Rule:

(i) Express $A = LDU$

(ii) From $LL^{-1} = I$ find L^{-1}

(iii) Find D^{-1}

(iv) From $UA^{-1} = D^{-1}L^{-1}$

Find $A^{-1} = ?$

In previous example we express $A = LDU$ where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 4/3 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Next find L^{-1}

$$LL^{-1} = L^{-1}L = I$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 4/3 & 1 \end{bmatrix} \begin{bmatrix} l_{11}' & l_{12}' & l_{13}' \\ l_{21}' & l_{22}' & l_{23}' \\ l_{31}' & l_{32}' & l_{33}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} l_{11}' & l_{12}' & l_{13}' \\ l_{11}' + l_{21}' & l_{12}' + l_{22}' & l_{13}' + l_{23}' \\ l_{11}' + \frac{4}{3} l_{21}' + l_{31}' & l_{12}' + \frac{4}{3} l_{22}' + l_{32}' & l_{13}' + \frac{4}{3} l_{23}' + l_{33}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$l_{11}' = 1$, $\quad l_{12}' = 0$, $\quad l_{13}' = 0$

$l_{11}' + l_{21}' = 0$, $\quad l_{12}' + l_{22} = 1$, $\quad l_{13}' + l_{23}' = 0$

$\Rightarrow l_{21} = -1$, $\quad \Rightarrow l_{22}' = 1$, $\quad l_{23}' = 0$

$l_{11}' + \frac{4}{3} l_{21}' + l_{31}' = 0$, $\quad l_{12}' + \frac{4}{3} l_{22}' + l_{32}' = 0$

$(1) + \frac{4}{3}(-1) + l_{31}' = 0$ $\quad \frac{4}{3}(1) + l_{32}' = 0$

$l_{31}' = +\frac{1}{3}$ $\quad \Rightarrow l_{32}' = -\frac{4}{3}$

$l_{13}' + \frac{4}{3} l_{23}' + l_{33}' = 1$

$0 + \frac{4}{3}(0) + l_{33}' = 1$

$\Rightarrow l_{33}' = 1$.

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1/3 & -4/3 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \Rightarrow D^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & -1/3 \end{bmatrix}$$

$$U A^{-1} = D^{-1} L^{-1}$$

$$\begin{bmatrix} 1 & - & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11}' & a_{12}' & a_{13}' \\ a_{21}' & a_{22}' & a_{23}' \\ a_{31}' & a_{32}' & a_{33}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & -1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ +1/3 & -4/3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a_{11}' - a_{21}' - 2a_{31}' & a_{12}' - a_{22}' - 2a_{32}' & a_{13}' - a_{23}' - 2a_{33}' \\ a_{21}' + a_{31}' & a_{22}' + a_{32}' & a_{23}' + a_{33}' \\ a_{31}' & a_{32}' & a_{33}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1/3 & 0 \\ -1/9 & +4/3 \cdot 3 & -1/3 \end{bmatrix}$$

$\Rightarrow a_{31}' = -1/9$, $a_{23}' + a_{33}' = 0$

$\Rightarrow a_{32}' = 4/9$, $\Rightarrow a_{23}' = 1/3$

$\Rightarrow a_{33} = -1/3$

$a_{22}' + a_{32}' = 1/3$

$\Rightarrow a_{22}' = \frac{1}{3} - \frac{4}{9} = -1/9$

$\Rightarrow a_{21}' = -2/9$, $a_{11}' = 5/9$, $a_{12}' = \frac{7}{9}$

and $a_{13} = -1/3$.

$$\Rightarrow A^{-1} = \begin{bmatrix} 5/9 & 7/9 & -1/3 \\ -2/9 & -1/9 & 1/3 \\ -1/9 & 4/9 & -1/3 \end{bmatrix}$$

(Answer)

INDIRECT METHOD (OR) Iterative Method

There are two methods to solve system of linear equations indirectly

(i) Jacobi Iterative Method

(ii) Gauss - Seidal Method.

(i) Jacobi- Iterative Method :-

Consider the linear system $Ax = b$.

Rearrange the equations so that diagonal elements are not zero and equation are diagonally dominent.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \rightarrow (i)$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \rightarrow (ii)$$
$$a_{31}x_1 + a_{32}x_2 + \dots + a_{3n}x_n = b_3 \rightarrow (iii)$$
$$\dots\dots\dots$$
$$\dots\dots\dots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \rightarrow (n)$$

Find x_1 from eq (i), x_2 from eq (ii), x_3 from eq (iii) and continuing from eq (n) find value of x_n.

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \dots - a_{1n}x_n)$$

$$x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \dots - a_{2n}x_n)$$

$$\dots\dots$$

$$x_n = \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \dots - a_{nn-1}x_{n-1})$$

In general we write it as

$$x_i = \frac{1}{a_{ii}}\left[b_i - \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}x_j\right], \; i = 1, 2, \dots, n$$

Setting iteration

$$x_1^{(k+1)} = 1/a_{11}\,[b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - \dots - a_{1n}x_n^{(k)}]$$
$$x_2^{(k+1)} = 1/a_{22}\,[b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k)} - \dots - a_{2n}x_n^{(k)}]$$
$$x_3^{(k+1)} = 1/a_{33}\,[b_3 - a_{31}x_1^{(k)} - a_{32}x_2^{(k)} - \dots - a_{3n}x_n^{(k)}]$$
$$\vdots$$
$$x_n^{(k+1)} = 1/a_{nn}\,[b_n - a_{n1}x_1^{(k)} - a_{n2}x_2^{(k)} - \dots - a_{nn-1}x_{n-1}^{(k)}]$$

In general we write it as setting iteration

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}x_j^{(k)}\right] ; \; i = 1, 2, 3, \dots, n$$

is the required formula for Jacobi's method. Starting from an initial guess, successive solution vector are computed untill convergence has been obtained.

* Condition of Convergence for a Linear System of equation (Convergence criteria for Jacobi's method)

The condition of convergence for system of linear equations is that linear system is diagonally dominant. The linear system of equations is said to be diagonally dominant if in each equation, the absolute value of diagonal coefficient is greater then or equal to the sum of absolute values of off-diagonal coefficients and is atleast one equation is actually greater then the sum i-e

$$|a_{ii}| \geq \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}| \quad \text{for all } i$$

and

$$|a_{ii}| > \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}| \quad \text{for atleast one value of } i$$

The condition is sufficient but not necessary condition for convergence. That is, if the coefficient satisfy this condition then method must converge. If not, the convergence is not guaranted. It may not and probably will not converge.

★ **Example:** Solve system of equations by Jacobi's Method

$10x_1 + 3x_2 + x_3 = 19 \longrightarrow (i)$

$3x_1 + 10x_2 + 2x_3 = 29 \longrightarrow (ii)$

$x_1 + 2x_2 + 10x_3 = 35 \longrightarrow (iii)$

Solution:- As equations are diagonally dominent

From (i) $x_1 = \frac{1}{10}(19 - 3x_2 - x_3)$

From (ii) $x_2 = \frac{1}{10}(29 - 3x_1 - 2x_3)$

From (iii) $x_3 = \frac{1}{10}(35 - x_1 - 2x_2)$

Setting iteration

$x_1^{(k+1)} = \frac{1}{10}(19 - 3x_2^{(k)} - x_3^{(k)})$

$x_2^{(k+1)} = \frac{1}{10}(29 - 3x_1^{(k)} - 2x_3^{(k)})$

$x_3^{(k+1)} = \frac{1}{10}(35 - x_1^{(k)} - 2x_2^{(k)})$

Let initial guess is

$x = (0, 0, 0)^T$

i.e $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0.$

1st Iteration:-

When $k = 0$

$x_1^{(1)} = \frac{1}{10}(19 - 3x_2^{(0)} - x_3^{(0)})$

$= \frac{1}{10}(19 - 3(0) - 0)$

$= 1.9$

$x_2^{(1)} = \frac{1}{10}(29 - 3x_1^{(0)} - 2x_3^{(0)})$

$= \frac{1}{10}(29 - 3(0) - 2(0))$

$= 2.9$

$x_3^{(1)} = \frac{1}{10}(35 - x_1^{(0)} - 2x_2^{(0)})$

$= \frac{1}{10}(35 - 0 - 2(0))$

$= 3.5$

2nd Iteration:- when $k = 1$

$x_1^{(2)} = \frac{1}{10}(19 - 3x_2^{(1)} - x_3^{(1)})$

$= \frac{1}{10}(19 - 3(2.9) - (3.5)) = 0.68$

$x_2^{(2)} = \frac{1}{10}(29 - 3x_1^{(1)} - 2x_3^{(1)})$

$= \frac{1}{10}(29 - 3(1.9) - 2(3.5)) = 1.63$

$x_3^{(2)} = \frac{1}{10}(35 - x_1^{(1)} - 2x_2^{(1)})$

$= \frac{1}{10}(35 - (1.9) - 2(2.9)) = 2.73$

3rd Iteration:

$k = 2$

$x_1^{(3)} = \frac{1}{10}(19 - 3x_2^{(2)} - x_3^{(2)})$

$= \frac{1}{10}(19 - 3(1.63) - 2.73) = 1.1380$

$x_2^{(3)} = \frac{1}{10}(29 - 3x_1^{(2)} - 2x_3^{(2)})$

$= \frac{1}{10}(29 - 3(0.68) - 2(2.73)) = 2.150$

$x_3^{(3)} = \frac{1}{10}(35 - x_1^{(2)} - 2x_2^{(2)})$

$= \frac{1}{10}(35 - 0.68 - 2(1.63))$

$= 3.106$

4th Iteration:

When $k = 3$

$$x_1^{(4)} = \frac{1}{10}\left(19 - 3x_2^{(3)} - x_3^{(3)}\right)$$
$$= \frac{1}{10}\left(19 - 3(2.150) - 2.106)\right)$$
$$= 0.9444$$

$$x_2^{(4)} = \frac{1}{10}\left(29 - 3x_1^{(3)} - 2x_3^{(3)}\right)$$
$$= \frac{1}{10}\left(29 - 3(1.133) - 2(3.106)\right)$$
$$= 1.9374$$

$$x_3^{(4)} = \frac{1}{10}\left(35 - x_1^{(3)} - 2x_2^{(3)}\right)$$
$$= \frac{1}{10}\left(35 - 1.133 - 2(2.150)\right)$$
$$= 2.9562$$

5th Iteration:

When $k = 4$

$$x_1^{(5)} = \frac{1}{10}\left(19 - 3x_2^{(4)} - x_3^{(4)}\right)$$
$$= \frac{1}{10}\left(19 - 3(1.9374) - 2.9562\right)$$
$$= 1.0232$$

$$x_2^{(5)} = \frac{1}{10}\left(29 - 3x_1^{(4)} - 2x_3^{(4)}\right)$$
$$= \frac{1}{10}\left(29 - 3(0.9444) - 2(2.9562)\right)$$
$$= 2.0254$$

$$x_3^{(5)} = \frac{1}{10}\left(35 - x_1^{(4)} - 2x_2^{(4)}\right)$$
$$= \frac{1}{10}\left(35 - 0.9444 - 2(1.9374)\right)$$
$$= 3.0181$$

6th Iteration:

When $k = 5$

$$x_1^{(6)} = \frac{1}{10}\left(19 - 3(2.0254) - 3.0181\right)$$
$$= 0.9906$$

$$x_2^{(6)} = \frac{1}{10}\left(29 - 3(1.0232) - 2(3.0181)\right)$$
$$= 1.9894$$

$$x_3^{(6)} = \frac{1}{10}\left(35 - 1.0232 - 2(2.0254)\right)$$
$$= 2.9926$$

7th Iteration:

When $k = 6$

$$x_1^{(7)} = \frac{1}{10}\left(19 - 3(1.9894) - 2.9926\right)$$
$$= 1.0039$$

$$x_2^{(7)} = \frac{1}{10}\left(29 - 3(0.9906) - 2(2.9926)\right)$$
$$= 2.0043$$

$$x_3^{(7)} = \frac{1}{10}\left(35 - 0.9906 - 2(1.9894)\right)$$
$$= 3.0031$$

8th Iteration:

When $k = 7$

$$x_1^{(8)} = \frac{1}{10}\left(19 - 3(2.0043) - 3.0031\right)$$
$$= 0.9984$$

$$x_2^{(8)} = \frac{1}{10}\left(29 - 3(1.0039) - 2(3.0031)\right)$$
$$= 1.9982$$

$$x_3^{(8)} = \frac{1}{10}\left(35 - 1.0039 - 2(2.0043)\right)$$
$$= 2.9988$$

9th Iteration:

When $k = 8$

$$x_1^{(9)} = \frac{1}{10}\left(19 - 3(1.9982) - 2.9988\right)$$
$$= 1.007$$

$x_2^{(9)} = \frac{1}{10}(29 - 3(0.9989) - 2(2.9986))$
$= 2.007$

$x_3^{(9)} = \frac{1}{10}(35 - 0.9984 - 2(1.9982))$
$= 3.0005$

10th Iteration :

when $k = 9$

$x_1^{(10)} = \frac{1}{10}(19 - 3(2.007) - 3.0005)$
$= 0.9979$

$x_2^{(10)} = \frac{1}{10}(29 - 3(1.007) - 2(3.0005))$
$= 1.9978$

$x_3^{(10)} = \frac{1}{10}(35 - 1.007 - 2(2.007))$
$= 2.9998$

11th Iteration :

When $k = 10$.

$x_1^{(11)} = \frac{1}{10}(19 - 3(1.9978) - 2.9998)$
$= 1.000$

$x_2^{(11)} = \frac{1}{10}(29 - 3(0.9979) - 2(2.9998))$
$= 2.000$

$x_3^{(11)} = \frac{1}{10}(35 - 0.9979 - 2(1.9978))$
$= 3.000$

Hence Solution vector is $x = [1, 2, 3]^T$

(Answer).

+ Example : Consider

$$4x - y + z = 7$$
$$4x - 8y + z = -21$$
$$-2x + y + 5z = 15$$

Solved by Jacobi's Method.

Solution : Re-write these equations we have

$x = \frac{1}{4}(7 + y - z)$

$y = \frac{1}{8}(21 + 4x + z)$

$z = \frac{1}{5}(15 + 2x - y)$

Setting iteration is

$x^{(k+1)} = 1/4(7 + y^{(k)} - z^{(k)})$

$y^{(k+1)} = 1/8(21 + 4x^{(k)} + z^{(k)})$

$z^{(k+1)} = 1/5(15 + 2x^{(k)} - y^{(k)})$

Let Guess vector is $(x^{(0)}, y^{(0)}, z^{(0)}) = (1, 2, 2)^T$

k	$x^{(k)}$	$y^{(k)}$	$z^{(k)}$
0	1	2	2
1	1.844	3.875	3.025
2	1.963	3.925	2.963
3	1.991	3.977	3.000
4	1.994	3.996	3.001
5	1.999	3.997	2.998
6	2.000	3.999	3.000
7	2.000	4.000	3.000

Hence Solution vector is

$(x, y, z)^T = (2, 4, 3)^T$

(Answer)

★ Gauss-Seidel Method:

Consider the linear system $Ax = b$. Re-arrange equations so that diagonal elements are not zero and equation are diagonally dominent.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + \cdots\cdots + a_{3n}x_n &= b_3 \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \quad & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots + a_{nn}x_n &= b_n \end{aligned}\right\} (i)$$

Find x_1 from 1^{st} equation, x_2 from 2^{nd} equation and so on find x_n. Then system (i) can be written as

$$x_1 = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n\right]$$

$$x_2 = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n\right]$$

$$\vdots$$

$$x_n = \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}x_{n-1}\right]$$

Setting iteration we get

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - \cdots - a_{1n}x_n^{(k)}\right]$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)} - \cdots - a_{2n}x_n^{(k)}\right]$$

$$x_3^{(k+1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)} - \cdots - a_{3n}x_n^{(k)}\right]$$

$$\vdots$$

$$x_n^{(k+1)} = \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1^{(k+1)} - a_{n2}x_2^{(k+1)} - \cdots - a_{n,n-1}x_{n-1}^{(k+1)}\right]$$

In general it also can be written as

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right], \quad i = 1, 2, 3, \ldots, n.$$

Starting from an initial guess, successive solution vectors are computed until convergence have been obtained.

Remark:-

Gauss-Seidel method converges twice as fast as Jacobi's Method.

* Example :-

Solve
$$10x_1 + 3x_2 + x_3 = 19$$
$$3x_1 + 10x_2 + 2x_3 = 29$$
$$x_1 + 2x_2 + 10x_3 = 35$$
by Gauss- Seidel Method.

Solution :-

Equations are diagonally dominent.

As
$$x_1 = \frac{1}{10}[19 - 3x_2 - x_3]$$
$$x_2 = \frac{1}{10}[29 - 3x_1 - 2x_3]$$
$$x_3 = \frac{1}{10}[35 - x_1 - 2x_2]$$

Setting Iteration are
$$x_1^{(k+1)} = 1/10[19 - 3x_2^{(k)} - x_3^{(k)}]$$
$$x_2^{(k+1)} = 1/10(29 - 3x_1^{(k+1)} - 2(x_3)^{(k)}]$$
$$x_3^{(k+1)} = \frac{1}{10}[35 - x_1^{(k+1)} - 2x_2^{(k+1)}]$$

Initial estimate $x^0 = (0,0,0)^T$.

1st Iteration: when $k=0$
$$x_1^{(1)} = \frac{1}{10}(19 - 3(0) - 0) = 1.9$$
$$x_2^{(1)} = \frac{1}{10}(29 - 3(0) - 2(0)) = 2.33$$
$$x_3^{(1)} = \frac{1}{10}(35 - 0 - 2(0)) = 2.844$$

2nd Iteration :

When $k=1$
$$x_1^{(2)} = \frac{1}{10}(19 - 3(2.33) - 2.844)$$
$$= 0.9166$$

$$x_2^{(2)} = \frac{1}{10}(29 - 3(0.9166) - 2(2.844)) = 2.0562$$
$$x_3^{(2)} = \frac{1}{10}(35 - 0.9166 - 2(2.0562)) = 2.9971$$

3rd Iteration :

when $k=2$
$$x_1^{(3)} = \frac{1}{10}(19 - 3(2.0562) - 2.9971) = 0.9834$$
$$x_2^{(3)} = \frac{1}{10}(29 - 3(0.9834) - 2(2.9971)) = 2.0056$$
$$x_3^{(3)} = \frac{1}{10}(35 - 0.9834 - 2(2.0056)) = 3.0005$$

4th Iteration :

When $k=3$
$$x_1^{(4)} = \frac{1}{10}(19 - 3(2.0056) - 3.0005) = 0.9983$$
$$x_2^{(4)} = \frac{1}{10}(29 - 3(0.9983) - 2(3.0005)) = 2.004$$
$$x_3^{(4)} = \frac{1}{10}(35 - 0.9983 - 2(2.0004)) = 3.0001$$

5th Iteration :

When $k=4$
$$x_1^{(5)} = \frac{1}{10}(19 - 3(2.004) - 3.0001) = 0.999$$
$$x_2^{(5)} = \frac{1}{10}(29 - 3(0.9999) - 2(3.0001)) = 2.000$$
$$x_3^{(5)} = \frac{1}{10}(35 - 0.999 - 2(2.000)) = 3.000$$

Hence solution vector is

$$x = [1, 2, 3]^T \quad \text{(Answer)}$$

Example:

Solve by Gauss Seidel method

$10x_1 + x_2 + 2x_3 = 44$

$x_1 + 2x_2 + 10x_3 = 61$

$2x_1 + 10x_2 + x_3 = 51$

Solution:

As equations are not diagonally dominant, rearranging we have

$10x_1 + x_2 + 2x_3 = 44$

$2x_1 + 10x_2 + x_3 = 51$

$x_1 + 2x_2 + 10x_3 = 61$

$\Rightarrow x_1 = \frac{1}{10}[44 - x_2 - 2x_3]$

$x_2 = \frac{1}{10}[51 - 2x_1 - x_3]$

$x_3 = \frac{1}{10}[61 - x_1 - 2x_2]$

Iterative equations are

$$x_1^{(k+1)} = \frac{1}{10}[44 - x_2^{(k)} - 2x_3^{(k)}]$$

$$x_2^{(k+1)} = \frac{1}{10}[51 - 2x_1^{(k+1)} - x_3^{(k)}]$$

$$x_3^{(k+1)} = \frac{1}{10}[61 - x_1^{(k+1)} - 2x_2^{(k+1)}]$$

Initial vector $\underline{x}^{(0)} = (0, 0, 0)^T$.

1st Iteration: When $k = 0$

$x_1^{(1)} = 1/10[44 - 0 - 0] = 4.4$

$x_2^{(1)} = 1/10[51 - 2(4.4) - 0] = 4.22$

$x_3^{(1)} = 1/10[61 - 4.4 - 2(4.22)] = 4.816$

2nd Iteration:

When $k = 1$

$x_1^{(2)} = 1/10[44 - 4.22 - 2(4.816)] = 3.0148$

$x_2^{(2)} = 1/10[51 - 2(3.0148) - 4.816] = 4.0154$

$x_3^{(2)} = 1/10[61 - 3.0148 - 2(4.0154)] = 4.9954$

3rd Iteration:

When $k = 2$

$x_1^{(3)} = 1/10[44 - 4.0154 - 2(4.9954)] = 2.9994$

$x_2^{(3)} = 1/10[51 - 2(2.9994) - 4.9954] = 4.0007$

$x_3^{(3)} = 1/10[61 - 2.9994 - 2(4.0007)] = 4.9999$

4th Iteration:

When $k = 3$

$x_1^{(4)} = 1/10[44 - 4.0007 - 2(4.9999)] = 2.9999$

$x_2^{(4)} = 1/10[51 - 2(2.9999) - 4.9999] = 4.0000$

$x_3^{(4)} = \frac{1}{10}[61 - 2.9999 - 2(4.0000)]$

$= 5.0000$

5^{th} Iteration :

when $k = 4$

$$x_1^{(5)} = \frac{1}{10}\left[44 - 4.000 - 2(5.000)\right]$$
$$= 3.000$$
$$x_2^{(5)} = \frac{1}{10}\left[51 - 2(3.000) - 5.000\right]$$
$$= 4.000$$
$$x_3^{(5)} = \frac{1}{10}\left[61 - 3.000 - 2(4.000)\right]$$
$$= 5.000$$

Hence Solution vector $x = (3, 4, 5)^T$

(Answer).

EXERCISE

1): Explain Jacobi's Method to solve iterative method.

2): Explain Gauss-Seidel method and write advatages of Gauss-Seidel method.

3): Write general iterative formula for both Jacobi's and Gauss-Seidel methods for solving a system of linear equation.

4): Describe the convergence criterion for a system of Linear equation.

5): Solve by Jacobi's Method carry out three iterations. Rearrange equation for Convergence

$$5x_1 + 2x_2 + x_3 = 14$$
$$x_1 + x_2 + 8x_3 = 20$$
$$x_1 + 5x_2 - x_3 = 10.$$

6): Apply 3 iteration of Gauss-Seidel method

$$8x_1 + x_2 - x_3 = 8$$
$$3x_1 - x_2 + 9x_3 = 12$$
$$x_1 - 7x_2 + 2x_3 = -4.$$

7). Apply 4 iteration of Gauss-Seidel method after arranging equations for convergence

$x_1 + 2x_2 + 5x_3 = 2$

$6x_1 - x_2 - x_3 = 3$

$2x_1 + 7x_2 - x_3 = -3$

8): Apply Jacobi's Method

$4x_1 + x_2 - x_3 = 0$

$x_1 - 4x_2 + 2x_3 = 13$

$2x_1 - x_2 + 5x_3 = 10.$

* Norm of a Matrix :-

If 'A' is an $n \times n$ matrix then norm of A is real number, denoted by $\|A\|$ s.t the following conditions are satisfied

(i) $\|A\| \geq 0$

(ii) $\|A\| = 0 \iff A = 0$

(iii) $\|K \cdot A\| = |K| \, \|A\|$

(iv) $\|A + B\| \leq \|A\| + \|B\|$

(v) $\|AB\| \leq \|A\| \, \|B\|$

Some Useful Norms :-

(i) Column Norm (OR) 1 - Norm

$$\|A\|_c = \|A\|_1 = \max_{j} \sum_{i=1}^{n} |a_{ij}|$$

Example :

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 8 & 6 \\ +1 & 3 & 9 \end{bmatrix}$$

$$\|A\|_1 = \text{Max}(1+3+1, \; 2+8+3, \; 5+6+9)$$

$$= \text{Max}(5, 13, 20)$$

$$= 20$$

$\|A\|_1 = 20.$

(ii) Row Norm (or) ∞- Norm.

$$\|A\|_r = \|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$$

Example: $\|A\|_r = \text{Max}\,(1+2+5,\ 3+8+6,\ 1+3+9)$

$= \text{Max}\,(8,\ 17,\ 13)$

$= 17$

(iii) Euclidian Norm (or) 2. Norm.

$$\|A\|_2 = \sqrt{\sum_{i,j} |a_{ij}|^2}$$

Example:

$$\|A\|_2 = \sqrt{|a_{11}|^2 + |a_{12}|^2 + |a_{13}|^2 + |a_{21}|^2 + |a_{22}|^2 + |a_{23}|^2 + |a_{31}|^2 + |a_{32}|^2 + |a_{33}|^2}$$

$= \sqrt{1+4+25+9+64+36+1+9+81}$

$= \sqrt{230}$

$= 15.1658$

* Condition Number :- If 'A' is a non-singular matrix then condition number of A is denoted and defined by

$$\text{Cond}(A) = K(A) = \|A\|\,\|A^{-1}\|.$$

Example :- If $A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 2 & 5 \\ 1 & -1 & 0 \end{bmatrix}$

then find $K_1(A)$, $K_2(A)$ and $K_\infty(A)$.

Solution :-

As $A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 2 & 5 \\ 1 & -1 & 0 \end{bmatrix}$ and

$$A^{-1} = \begin{bmatrix} 1 & -1/5 & -2/5 \\ 1 & -1/5 & -7/5 \\ -1 & 2/5 & 4/5 \end{bmatrix}$$

(i) $\|A\|_1 = \text{Max}\,(2+3+1,\ 0+2+1,\ 1+5+0)$

$= \text{Max}\,(6,\ 3,\ 6)$

$= 6$

$\|A^{-1}\|_1 = \text{Max}\,\left(1+1+1,\ \frac{1}{5}+\frac{1}{5}+\frac{2}{5},\ \frac{2}{5}+\frac{7}{5}+\frac{4}{5}\right)$

$= \text{Max}\,(3,\ 4/5,\ 13/5)$

$= 3$

$K_1(A) = \|A\|_1 \cdot \|A^{-1}\|_1$

$= 6 \times 3 = 18$

(ii) $\|A\|_\infty = \text{Max}(2+0+1,\ 3+2+5,\ 1+1+0)$

$= \text{Max}(3,\ 10,\ 2)$

$= 10$

$\|A^{-1}\|_\infty = \text{Max}\left(1+\frac{1}{5}+\frac{2}{5},\ 1+\frac{1}{5}+\frac{7}{5},\ 1+\frac{2}{5}+\frac{4}{5}\right)$

$= \text{Max}(1.600,\ 2.600,\ 2.200)$

$= 2.6000$

$K_\infty(A) = \|A\|_\infty \cdot \|A^{-1}\|_\infty$

$= 10 \times 2.6000$

$= 26.000$

(iii) $\|A\|_2 = \sqrt{\sum_{i,j} |a_{ij}|^2}$

$= \sqrt{4+0+1+9+4+25+1+1+0}$

$= \sqrt{45}$

$\|A^{-1}\|_2 = \sqrt{1+\frac{1}{25}+\frac{4}{25}+1+\frac{1}{25}+\frac{49}{25}+1+\frac{4}{25}+\frac{16}{25}}$

$= \sqrt{3+\frac{75}{25}}$

$= \sqrt{6}$

$K_2(A) = \|A\|_2 \cdot \|A^{-1}\|_2$

$= \sqrt{45} \cdot \sqrt{6}$

$= 3\sqrt{30}$

$= 16.4317.$

Theorem:- Condition number of a matrix is always greater then or equal to one.

Proof:- Let A be any square matrix

then
$A \cdot A^{-1} = I$

Now $1 \le \|I\|$

$= \|A \cdot A^{-1}\|$

$\le \|A\| \cdot \|A^{-1}\|$

$= \text{Cond}(A)$

$\Rightarrow 1 \le \text{Cond}(A)$

or $\text{Cond}(A) \ge 1.$

(proved).

* **Importance of Condition Number:-** Condition number of matrix is a measure of condition of the matrix. Large condition number of matrix indicates ill condition of the System of linear equation $Ax = b$. Small condition number of matrix A indicates the system is well condition or Stable.

Example :-

Consider the system

$1.01x_1 + 0.99x_2 = 2$

$0.99x_1 + 1.01x_2 = 2$

then $x = [1, 1]^T$ is its solution but if it is written as

$1.01x_1 + 0.99x_2 = 2$

$0.99x_1 + 1.01x_2 = -2$

then $x = [100, -100]^T$ is its solution. Determin it is stable or ill condition.

Solution :-

Here $A = \begin{bmatrix} 1.01 & 0.99 \\ 0.99 & 1.01 \end{bmatrix}$

$|A| = 1.0201 - 0.9801$

$|A| = 0.0400$

$A^{-1} = \begin{bmatrix} 25.250 & -24.75 \\ -24.75 & 25.25 \end{bmatrix}$

$\|A\|_1 = Max(1.01 + 0.99, 0.99 + 1.01)$

$= Max(2, 2)$

$= 2$

$\|A^{-1}\|_1 = Max(25.25 + 24.75, 24.75 + 25.25)$

$= Max(50, 50)$

$= 50$

$K_1(A) = \|A\|_1 \cdot \|A^{-1}\|_1$

$= 2 \times 50 = 100$

And

$\|A\|_\infty = Max(1.01 + 0.99, 0.99 + 1.01)$

$= Max(2, 2)$

$= 2$

$\|A^{-1}\|_\infty = Max(25.25 + 24.75, 24.75 + 25.25)$

$= 50$

$K_\infty(A) = \|A\|_\infty \cdot \|A^{-1}\|_\infty$

$= 2 \times 50$

$= 100$

Condition number is large so system is ill condition.

Remark :-

A system is supposed to be ill-condition when one or more of following indicators are presents

(1) Small change in coefficients of matrix A or in element of b results a large change in solution vector.

(2) Matrix A possesses a very small determinent as compared to the average of its elements.

(3) If $Cond(A) = \|A\|\|A^{-1}\|$ is large.

* Error & Residual of an Approximation :-

Consider the linear system ...

$$A\underline{x} = \underline{b}$$

Let x be the exact solution and $x^{(a)}$ be approximate solution of system then

$$e = x - x^{(a)}$$

is called error and

$$r = A\underline{x} - A\underline{x}^{(a)}$$

is called residual of an approximation. The quantities $\frac{\|r\|}{\|b\|}$ and $\frac{\|e\|}{\|x\|}$ are called relative residual and relative error of the approximation solution $\underline{x}^{(a)}$.

* *Example* :

Let $1.01x_1 + 0.99x_2 = 2$

$0.99x_1 + 1.01x_2 = 2$

then $\underline{x} = [1, 1]^T$ is exact solution and $\underline{x} = [1.01, 1.01]^T$ is approximate solution then find error and residual.

Solution:- $\underline{e} = \underline{x} - \underline{x}^{(a)}$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1.01 \\ 1.01 \end{bmatrix}$$

$$= \begin{bmatrix} -0.01 \\ -0.01 \end{bmatrix}$$

And $\underline{r} = A\underline{x} - A\underline{x}^{(a)}$

$$\Rightarrow r = b - A\underline{x}^{(a)}$$

$$= \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 2.02 \\ 2.02 \end{bmatrix}$$

$$= \begin{bmatrix} -0.02 \\ -0.02 \end{bmatrix}$$

Here e and r both are small Now if we take

$$\underline{x}^{(a)} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

then $\underline{e} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\underline{r} = \begin{bmatrix} -0.02 \\ 0.02 \end{bmatrix}$

Here error is relatively large to the residual but, as the system is ill condition, so all this indicates that error and residual does not indicate whether the system is ill condition or well condition (stable).

An Important Result :-

Now consider $A\underline{x} = b$, then

$$e = \underline{x} - \underline{x}^{(a)}$$

$\because x^{(a)}$ is approximate Solution

and

$$\underline{r} = A\underline{x} - A\underline{x}^{(a)}$$

$$= A(\underline{x} - \underline{x}^{(a)})$$

$$r = Ae \rightarrow (i)$$

$$\Rightarrow e = A^{-1} r \rightarrow (ii)$$

From (i)

$$\|r\| = \|Ae\|$$
$$\leq \|A\| \|e\| \rightarrow (iii)$$

From (ii)

$$\|e\| = \|A^{-1} r\|$$
$$\leq \|A^{-1}\| \|r\| \rightarrow (iv)$$

From (iii)

$$\frac{\|r\|}{\|A\|} \leq \|e\| \rightarrow (v)$$

From (iv) and (v)

$$\frac{\|r\|}{\|A\|} \leq \|e\| \leq \|A^{-1}\| \|r\| \rightarrow (vi)$$

Now if we choose $x^{(a)} = 0$
then

$$r = Ax - Ax^{(a)}$$
$$= Ax - 0$$
$$= Ax$$
$$r = b$$

and

$$e = x - x^{(a)}$$
$$e = x$$

Then (vi) becomes

$$\frac{\|b\|}{\|A\|} \leq \|x\| \leq \|A^{-1}\| \|b\|$$

$$\Rightarrow \frac{\|A\|}{\|b\|} \geq \frac{1}{\|x\|} \geq \frac{1}{\|A^{-1}\| \|b\|}$$

$$\Rightarrow \frac{1}{\|A^{-1}\| \|b\|} \leq \frac{1}{\|x\|} \leq \frac{\|A\|}{\|b\|} \rightarrow (vii)$$

Multiplying (vi) by (vii)

$$\frac{1}{\|A\| \|A^{-1}\|} \cdot \frac{\|r\|}{\|b\|} \leq \frac{\|e\|}{\|x\|} \leq \|A\| \|A^{-1}\| \frac{\|r\|}{\|b\|}$$

$$\Rightarrow \frac{1}{K(A)} \frac{\|r\|}{\|b\|} \leq \frac{\|e\|}{\|x\|} \leq K(A) \cdot \frac{\|r\|}{\|b\|} \rightarrow (viii)$$

Now if $K(A) \approx 1$ i.e system is stable, then from (viii)

$$\frac{\|r\|}{\|b\|} \leq \frac{\|e\|}{\|x\|} \leq \frac{\|r\|}{\|b\|}$$

$$\Rightarrow \frac{\|r\|}{\|b\|} \approx \frac{\|e\|}{\|x\|}$$

Hence if the system is stable then relative residual and relative error are approximately equal.

SORRY

Eigen Values & Eigen Vectors :-

If 'A' is a square matrix over $\mathbb{R}$, then $\lambda \in \mathbb{R}$ is said to be eigen value of A if for some non-zero column vector

$$A\underline{v} = \lambda \underline{v}$$

Example:

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\Rightarrow 5 = \lambda$ is eigen value of that matrix.

Remark:-

If $A\underline{v} = \lambda \underline{v}$, $v \neq 0$, then λ is called eigen value of A and $\underline{v}$ is called eigen vector of A corresponding to λ. An eigen value is also called characteristic root or latent root and eigen vector is also called characteristic vector or latent vector.

* **Theorem:-**
 If A is invertible and $AB=0$ then $B=0$.

Proof:- As A is invertible, so A^{-1} exists.
Thus $AB=0$
$\Rightarrow A^{-1}(AB)=A^{-1}0$
$\Rightarrow (A^{-1}A)B=0$
$\Rightarrow IB=0$
$\Rightarrow B=0$
(proved)

* **Remark:-**
(i) From above theorem if $B\neq 0$ then A is non-invertible.
(ii) Now
$A\underline{v}=\lambda\underline{v}$
$\Rightarrow A\underline{v}-\lambda\underline{v}=0$
$\Rightarrow (A-\lambda I)\underline{v}=0$
$\Rightarrow A-\lambda I$ is singular $\because v\neq 0$
$\Rightarrow |A-\lambda I|=0$.

* **POWER METHOD:-**
It is an iterative method for finding dominant eigen value and eigen vector of a matrix.

(i) Dominant Eigen Value:-
An eigen value λ of square matrix A is said to be dominant eigen value if $|\lambda|$ has largest value as compared to absolute value of the other eigen value of A. Then corresponding eigen vector is called dominant eigen vector.

(ii) Normalized Vector:-
An eigen vector $\underline{x}$ is said to be normalized if the co-ordinate of largest magnitude is 1.
For example
$[1,0,0]^T$, $[1,1,1]^T$, $[0,1,1]^T$ are all normalized vectors.

WORKING RULE FOR POWER METHOD:-
To find dominant eigen value and corresponding eigen vector of matrix take x_0 is initial vector which is normalized vector like $x_0=[1,1,1]^T$

$A\underline{x}_0 = C_1 \underline{x}_1$ where x_1 is normalized
$A x_1 = C_2 \underline{x}_2$ where x_2 is normalized
$A \underline{x}_2 = C_3 \underline{x}_3$ where x_3 is normalized
$\vdots \quad \vdots$

$A \underline{x}_n = C_{n+1} \underline{x}_{n+1}$ where x_{n+1} is normalized
then $\lambda = C_{n+1}$ is dominant eigen value and $\underline{x}_{n+1}$ is the corresponding eigen vector.
The iterative process will be stopped when $\underline{x}_{n+1} = \underline{x}_n$.

Working Rule for finding absolute least eigen value :-

Step I: If λ_1 is the dominant eigen value and x_1 be corresponding dominant eigen vector, then find
$B = A - \lambda_1 I$

Step II: Find dominant eigen value of B say it is λ_2. (By taking negative common).

Step III: Absolute least eigen value of
$A = \lambda_1 + \lambda_2$.

Example :-
Find dominant and absolute least eigen pair of $A = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}$

Solution :-
Taking initial guess vector
$x_0 = [1 \quad 1]^T$

$$A\underline{x}_0 = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \end{bmatrix} = 10\begin{bmatrix} 0.7 \\ 1 \end{bmatrix} = \lambda_1 \underline{x}_1$$

$$A x_1 = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} 0.7 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.5 \\ 9.4 \end{bmatrix} = 9.4\begin{bmatrix} 0.585 \\ 1 \end{bmatrix} = \lambda_2 \underline{x}_2$$

$$A \underline{x}_2 = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} 0.585 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.925 \\ 9.17 \end{bmatrix} = 9.17\begin{bmatrix} 0.537 \\ 1 \end{bmatrix} = \lambda_3 \underline{x}_3$$

$$A \underline{x}_3 = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} 0.531 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.605 \\ 9.074 \end{bmatrix} = 9.074\begin{bmatrix} 0.516 \\ 1 \end{bmatrix} = \lambda_4 \underline{x}_4$$

$$A \underline{x}_4 = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} 0.516 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.581 \\ 9.032 \end{bmatrix} = 9.032\begin{bmatrix} 0.507 \\ 1 \end{bmatrix} = \lambda_5 \underline{x}_5$$

$$A \underline{x}_5 = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} 0.507 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.535 \\ 9.014 \end{bmatrix} = 9.014\begin{bmatrix} 0.503 \\ 1 \end{bmatrix} = \lambda_6 \underline{x}_6$$

$$A \underline{x}_6 = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} 0.503 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.515 \\ 9.006 \end{bmatrix} = 9.006\begin{bmatrix} 0.5013 \\ 1 \end{bmatrix} = \lambda_7 \underline{x}_7$$

$$A \underline{x}_7 = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} 0.5013 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.506 \\ 9.0026 \end{bmatrix} = 9.0026\begin{bmatrix} 0.500 \\ 1 \end{bmatrix} = \lambda_8 \underline{x}_8$$

$A x_3 = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.5 \\ 9 \end{bmatrix} = 9 \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = \lambda_4 x_4$

we see that $x_3 = x_4$. So stop iteration and $\lambda = 9$ is dominant eigen value and $[0.5, 1]^T$ is dominant eigen vector.

For absolute least eigen value calculate

$B = A - \lambda I$

$\because \lambda$ is dominant eigen value.

$B = A - 9I$

$= \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$

$= \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$

Guess vector $x_0 = [1 \ 1]^T$

$B x_0 = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \lambda_1 x_1$

$B x_1 = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \begin{bmatrix} -5 \\ 2.5 \end{bmatrix} = -5 \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \lambda_2 x_2$

$B x_2 = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \begin{bmatrix} -5 \\ 2.5 \end{bmatrix} = -5 \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \lambda_3 x_3$

So $\lambda = -5$ is dominant eigen value of B.

Absolute least eigen value of $A = \lambda_1 + \lambda_2$

$= 9 + (-5)$

$= 4.$

(Answer).

* Example:- Using power method to find the dominant and smallest eigen value and corresponding eigen vector of matrix

$A = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ by take $y_0 = [1, 1, 1]^T$

Solution:

$A y_0 = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 0.6 \\ -0.2 \end{bmatrix} = C_1 y_1$

$A y_1 = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.6 \\ -0.2 \end{bmatrix} = \begin{bmatrix} 4.6 \\ 1.03 \\ 0.20 \end{bmatrix} = 4.6 \begin{bmatrix} 1 \\ 0.22 \\ 0.04 \end{bmatrix} = C_2 y_2$

$A y_2 = A \begin{bmatrix} 1 \\ 0.22 \\ 0.04 \end{bmatrix} = \begin{bmatrix} 4.22 \\ 0.48 \\ -0.04 \end{bmatrix} = 4.22 \begin{bmatrix} 1 \\ 0.11 \\ -0.01 \end{bmatrix} = C_3 y_3$

$A y_3 = A \begin{bmatrix} 1 \\ 0.11 \\ -0.01 \end{bmatrix} = \begin{bmatrix} 4.11 \\ 0.21 \\ 0.01 \end{bmatrix} = 4.11 \begin{bmatrix} 1 \\ 0.05 \\ 0.00 \end{bmatrix} = C_4 y_4$

$A y_4 = A \begin{bmatrix} 1 \\ 0.05 \\ 0.00 \end{bmatrix} = \begin{bmatrix} 4.05 \\ 0.1 \\ 0 \end{bmatrix} = 4.05 \begin{bmatrix} 1 \\ 0.03 \\ 0 \end{bmatrix} = C_5 y_5$

$A y_5 = A \begin{bmatrix} 1 \\ 0.03 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.03 \\ 0.06 \\ 0 \end{bmatrix} = 4.03 \begin{bmatrix} 1 \\ 0.02 \\ 0 \end{bmatrix} = C_6 y_6$

$$Ay_6 = A\begin{bmatrix}1\\0.02\\0\end{bmatrix} = \begin{bmatrix}4.02\\0.04\\0\end{bmatrix} = 4.02\begin{bmatrix}1\\0.01\\0\end{bmatrix} = C_7 y_7$$

$$Ay_7 = A\begin{bmatrix}1\\0.01\\0\end{bmatrix} = \begin{bmatrix}4.01\\0.02\\0\end{bmatrix} = 4.01\begin{bmatrix}1\\0\\0\end{bmatrix} = C_8 y_8$$

$$Ay_8 = A\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}4\\0\\0\end{bmatrix} = 4\begin{bmatrix}1\\0\\0\end{bmatrix} = C_8 = C_9$$

$\lambda_1 = 4$ is a dominant eigen value and corresponding eigen vector $[1, 0, 0]^T$.

For a smallest eigen value calculate

$B = A - \lambda I$

$\because \lambda$ is dominant eigen value.

$$B = A - 4I$$

$$= \begin{bmatrix}4 & 1 & 0\\0 & 2 & 1\\0 & 0 & -1\end{bmatrix} - 4\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\0 & -2 & 1\\0 & 0 & -5\end{bmatrix}$$

Let $x_0 = (1, 1, 1)^T$

$$Bx_0 = \begin{bmatrix}0 & 1 & 0\\0 & -2 & 1\\0 & 0 & -5\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\-1\\-5\end{bmatrix} = -5\begin{bmatrix}-0.2\\0.2\\1\end{bmatrix} = C_1 x_1$$

$$Bx_1 = B\begin{bmatrix}-0.2\\0.2\\1\end{bmatrix} = \begin{bmatrix}0.2\\0.6\\5\end{bmatrix} = -5\begin{bmatrix}-0.04\\-0.12\\1\end{bmatrix} = C_2 x_2$$

$$Bx_2 = B\begin{bmatrix}-0.04\\-0.12\\1\end{bmatrix} = \begin{bmatrix}-0.12\\1.24\\-5\end{bmatrix} = -5\begin{bmatrix}0.02\\-0.25\\1\end{bmatrix} = C_3 x_3$$

$$Bx_3 = B\begin{bmatrix}0.02\\-0.25\\1\end{bmatrix} = \begin{bmatrix}-0.25\\1.5\\-5\end{bmatrix} = -5\begin{bmatrix}0.05\\-0.3\\1\end{bmatrix} = C_4 x_4$$

$$Bx_4 = B\begin{bmatrix}0.05\\-0.3\\1\end{bmatrix} = \begin{bmatrix}-0.3\\1.06\\-5\end{bmatrix} = -5\begin{bmatrix}0.06\\-0.32\\1\end{bmatrix} = C_5 x_5$$

$\lambda_2 = -5$ is dominant eigen value of B.

Smallest eigen value of $A = \lambda_1 + \lambda_2$

$= 4 + (-5)$

$= -1$

Corresponding eigen vector is $(0.06, -0.32, 1)^T$

(Answer)

* *Example:-*

Use power Method to compute the largest eigenvalue and corresponding eigen vector of matrix

$$\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$$

Solution:- Let $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$

Take $y_0 = [1 \ 0 \ 0]^T$ as initial vector
Then by Power method

$$Ay_0 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0.5 \\ -0.50 \end{bmatrix}$$

$$Ay_1 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -4 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0.80 \\ -0.80 \end{bmatrix}$$

$$Ay_2 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.80 \\ -0.80 \end{bmatrix} = \begin{bmatrix} 5.60 \\ 5.20 \\ -5.20 \end{bmatrix} = 5.60\begin{bmatrix} 1 \\ 0.93 \\ -0.93 \end{bmatrix}$$

$$Ay_3 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.93 \\ -0.93 \end{bmatrix} = \begin{bmatrix} 5.86 \\ 5.72 \\ -5.72 \end{bmatrix} = 5.86\begin{bmatrix} 1 \\ 0.98 \\ -0.98 \end{bmatrix}$$

$$Ay_4 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.98 \\ -0.98 \end{bmatrix} = \begin{bmatrix} 5.96 \\ 5.92 \\ -5.92 \end{bmatrix} = 5.96\begin{bmatrix} 1 \\ 0.99 \\ -0.99 \end{bmatrix}$$

$$Ay_5 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.99 \\ -0.99 \end{bmatrix} = \begin{bmatrix} 5.98 \\ 5.96 \\ -5.96 \end{bmatrix} = 5.98\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$Ay_6 = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ -6 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

So dominant or largest eigen value is $\lambda = 6$ and corresponding eigen vector is $[1, 1, -1]^T$

(Answer).

Linear Operators

An operation P is said to be Linear if for all f_1, f_2 and α_1, α_2

$$P(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 P(f_1) + \alpha_2 P(f_2)$$

where P is defined on space f of functions and $f_1, f_2 \in f$ and $\alpha_1, \alpha_2 \in R$.

In numerical analysis the following linear operators are defined

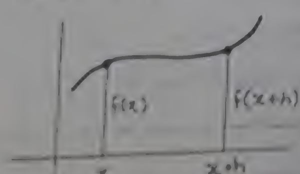
(i) Shift Operator
(ii) Forward difference Operator
(iii) Backward difference Operator.
(iv) Central difference Operator
(v) Average or Mean value Operator.

(i) Shift Operator :-

$$\boxed{E f(x) = f(x+h)}$$

and
$$E^2 f(x) = E(f(x+h))$$
$$= f(x+h+h)$$
$$E^2 f(x) = f(x+2h)$$

Similarly; In general
$$E^n f(x) = f(x+nh)$$



(ii) Forward Difference Operator:-

Let the values $x_0, x_0+h, x_0+2h, \dots, x_0+nh$ be corresponding independent variables and $f(x_0), f(x_0+h), f(x_0+2h), \dots, f(x_0+nh)$ be corresponding dependent variable of function $f(x)$.
The values of x are known as argument and $f(x)$ is called entries.
where h be a common difference between intervals.
Now Forward difference operator is defined as

$$\Delta f(x) = f(x+h) - f(x)$$

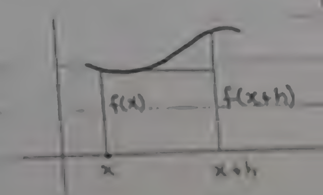
which is of order one.

Similarly; $\Delta f(x+h) = f(x+2h) - f(x+h)$
$$\Delta f(x+nh) = f(x+(n+1)h) - f(x+nh)$$

In 2nd order forward difference defined as

$$\Delta^2 f(x) = \Delta[f(x+h) - f(x)]$$
$$= f(x+2h) - f(x+h) - [f(x+h) - f(x)]$$
$$= f(x+2h) - f(x+h) - f(x+h) + f(x)$$
$$= f(x+2h) - 2f(x+h) + f(x).$$

Similarly; $\Delta^3 f(x) = f(x+3h) - 3f(x+2h) + 3f(x+h) + f(x).$



(iii) Backward Difference operator:-

It is denoted and defined as

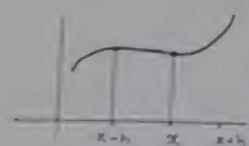
$$\nabla f(x) = f(x) - f(x-h)$$

and

$$\nabla f(x+h) = f(x+h) - f(x)$$

$$\nabla f(x+2h) = f(x+2h) - f(x+h)$$

$$\nabla f(x+rh) = f(x+rh) - f(x+(r-1)h)$$



(iv) Centeral Difference Operator:-

The first order centeral difference operator is denoted and defined as

$$\delta f(x) = f(x+\tfrac{h}{2}) - f(x-\tfrac{h}{2})$$

(v) Average Operator:-

Average operator denoted by μ and defined by

$$\mu f(x) = \frac{f(x+h/2) + f(x-h/2)}{2}$$

* RELATIONS WITH SHIFT OPERATOR:-

(i) Relation between Forward difference operator and Shift Operator:-

$$\Delta f(x) = f(x+h) - f(x)$$

$$= E f(x) - f(x)$$

$$= (E-1) f(x)$$

$$\Rightarrow \Delta f(x) = (E-1) f(x)$$

$$\Rightarrow \Delta = E - 1$$

$$\Rightarrow E = \Delta + 1.$$

(ii) Relation between Backward difference operator and Shift Operator:-

$$\nabla f(x) = f(x) - f(x-h)$$

$$\nabla f(x) = f(x) - E^{-1} f(x)$$

$$= (1 - E^{-1}) f(x)$$

$$\nabla f(x) = (1 - E^{-1}) f(x)$$

$$\Rightarrow \nabla = (1 - E^{-1})$$

$$\Rightarrow E^{-1} = 1 - \nabla$$

$$\Rightarrow E = (1-\nabla)^{-1}$$

(iii) Relation between Centeral difference operator and Shift operator:

$$\delta f(x) = f(x + h/2) - f(x - h/2)$$

$$= E^{1/2} f(x) - E^{-1/2} f(x)$$

$$\delta f(x) = (E^{1/2} - E^{-1/2}) f(x)$$

$$\delta = E^{1/2} - E^{-1/2}$$

$$\delta^2 = (E^{1/2} - E^{-1/2})^2$$

$\Rightarrow \delta^2 = E + E^{-1} - 2$

Relation between Average Operator and Shift Operator:

$$\mu f(x) = \frac{f(x+h/2) + f(x-h/2)}{2}$$

$$= \frac{E^{1/2} f(x) + E^{-1/2} f(x)}{2}$$

$$\mu f(x) = \frac{(E^{1/2} + E^{-1/2}) f(x)}{2}$$

$$\mu = \frac{E^{1/2} + E^{-1/2}}{2}$$

Formulae :

$\Delta = E - 1$

$E = 1 + \Delta$

$\nabla = 1 - E^{-1}$

$E = (1 - \nabla)^{-1}$

$\delta = E^{1/2} - E^{-1/2}$

$\mu = \frac{1}{2}[E^{1/2} + E^{-1/2}]$

* **Question:-** Prove the following Identities

(i) $E = e^{hD}$; $D = d/dx$

$E f(x) = f(x+h)$

$= f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots$

$= f(x) + hD f(x) + \frac{h^2}{2!} D^2 f(x) + \frac{h^3}{3!} D^3 f(x) + \cdots$

$E f(x) = \left(1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \cdots\right) f(x)$

$\Rightarrow E = 1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \cdots$

$E = e^{hD}$ where $D = \frac{d}{dx}$. (proved).

(ii) $\Delta \sin k = 2\sin(h/2)\cos(k + h/2)$.

L.H.S $\Delta \sin k$

$= \sin(k+h) - \sin k$

$= 2\cos\left(\frac{k+h+k}{2}\right)\sin\left(\frac{k+h-k}{2}\right)$

$= 2\cos\left(\frac{2k+h}{2}\right)\sin(h/2)$

$= 2\sin(h/2)\cos(k + h/2)$ (proved).

(iii) $\Delta - \nabla = \Delta\nabla = \nabla\Delta = \delta^2$

$\Delta\nabla = (E-1)(1-E^{-1})$

$= E - 1 - 1 + E^{-1} = E - 2 + E^{-1} \rightarrow$ (i)

$\Delta - \nabla = (E-1) - (1 - E^{-1})$

$= E - 1 - 1 + E^{-1}$

$= E - 2 + E^{-1} \longrightarrow$ (ii)

$\Rightarrow \delta^2 = E + E^{-1} - 2$

(iv) Relation between Average Operator and Shift Operator:

$$\mu f(x) = \frac{f(x+h/2) + f(x-h/2)}{2}$$

$$= \frac{E^{1/2} f(x) + E^{-1/2} f(x)}{2}$$

$$\mu f(x) = \frac{(E^{1/2} + E^{-1/2}) f(x)}{2}$$

$$\mu = \frac{E^{1/2} + E^{-1/2}}{2}$$

Formulae:

$\Delta = E - 1$

$E = 1 + \Delta$

$\nabla = 1 - E^{-1}$

$E = (1-\nabla)^{-1}$

$\delta = E^{1/2} - E^{-1/2}$

$\mu = \frac{1}{2}[E^{1/2} + E^{-1/2}]$

* Question:- Prove the following Identities

(i) $E = e^{hD}$; $D = d/dx$.

$E f(x) = f(x+h)$

$= f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots$

$= f(x) + hD f(x) + \frac{h^2}{2!} D^2 f(x) + \frac{h^3}{3!} D^3 f(x) + \cdots$

$E f(x) = \left(1 + hD + \frac{h^2 D^2}{2!} + \frac{h^3 D^3}{3!} + \cdots\right) f(x)$

$\Rightarrow E = 1 + hD + \frac{h^2 D^2}{2!} + \frac{h^3 D^3}{3!} + \cdots$

$E = e^{hD}$ where $D = \frac{d}{dx}$. (proved).

(ii) $\Delta \sin k = 2\sin(h/2)\cos(k + h/2)$.

L.H.S $\Delta \sin k$

$= \sin(k+h) - \sin k$

$= 2\cos\left(\frac{k+h+k}{2}\right)\sin\left(\frac{k+h-k}{2}\right)$

$= 2\cos\left(\frac{2k+h}{2}\right)\sin(h/2)$

$= 2\sin(h/2)\cos(k + h/2)$ (proved).

(iii) $\Delta - \nabla = \Delta\nabla = \nabla\Delta = \delta^2$

$\Delta\nabla = (E-1)(1-E^{-1})$

$= E - 1 - 1 + E^{-1} = E - 2 + E^{-1} \rightarrow$ (i)

$\Delta - \nabla = (E-1) - (1 - E^{-1})$

$= E - 1 - 1 + E^{-1}$

$= E - 2 + E^{-1} \longrightarrow$ (ii)

$\nabla\Delta = (1 - E^{-1})(E - 1)$
$= E - 1 - 1 + E^{-1}$
$= E - 2 + E^{-1} \longrightarrow$ (iii)
$\delta^2 = (E^{1/2} - E^{-1/2})^2$
$= E + E^{-1} - 2 \longrightarrow$ (iv)
From (i), (ii), (iii) and (iv) (proved).

(iv) $\sum_{k=0}^{n-1} \Delta^2 f_k = \Delta f_n - \Delta f_0$

$\Delta^2 f_k = \Delta(\Delta f_k)$
$= \Delta(f_{k+1} - f_k)$
$\Delta^2 f_k = \Delta f_{k+1} - \Delta f_k \longrightarrow$ (i)

Put $k = 0, 1, 2, \ldots, n-1$ in eq (i)

$\Delta^2 f_0 = \Delta f_1 - \Delta f_0$
$\Delta^2 f_1 = \Delta f_2 - \Delta f_1$
$\Delta^2 f_2 = \Delta f_3 - \Delta f_2$
$\vdots$
$\Delta^2 f_{n-1} = \Delta f_n - \Delta f_{n-1}$

Adding
$\sum_{k=0}^{n-1} \Delta^2 f_k = \Delta f_n - \Delta f_0$ (proved).

(v) $\Delta^n e^{ax+b} = e^{ax+b}(e^{ah} - 1)^n$

$\Delta e^{ax+b} = e^{a(x+h)+b} - e^{ax+b}$
$= e^{ax+ah+b} - e^{ax+b}$
$= e^{ax+b}(e^{ah} - 1) \longrightarrow$ (i)

$\Delta^2 e^{ax+b} = \Delta(\Delta e^{ax+b})$
$= \Delta(e^{ax+b}(e^{ah} - 1))$ ∵ From (i)

$= (e^{ah} - 1)(\Delta e^{ax+b})$
$(e^{ah} - 1)\{e^{ax+b}(e^{ah} - 1)\}$
$e^{ax+b}(e^{ah} - 1)^2$

Continuing in this way we have

$\Delta^n e^{ax+b} = e^{ax+b}(e^{ah} - 1)^n$ (proved)

(vi) $\frac{\Delta^2}{E}(\sin x) = 2\sin x(\cosh - 1)$.

Taking L.H.S

$\frac{\Delta^2}{E}(\sin x) = \Delta^2 E^{-1}(\sin x)$
$= \Delta^2 \sin(x - h)$
$= \Delta(\Delta \sin(x - h))$
$= \Delta\{\sin x - \sin(x - h)\}$
$= \Delta \sin x - \Delta \sin(x - h)$
$= \sin(x+h) - \sin x - \sin x + \sin(x - h)$
$= \sin(x+h) - 2\sin x + \sin(x-h)$
$2\sin\left(\frac{x+h+x-h}{2}\right)\cos\left(\frac{x+h-x+h}{2}\right) - 2\sin x$

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$= 2\sin x \cosh - 2\sin x$
$= 2\sin x(\cosh - 1)$
$=$ R.H.S (proved).

(vii) $\Delta(f_k g_k) = f_{k+1}\Delta g_k + g_k \Delta f_k$.

L.H.S $\Delta(f_k g_k)$
$= f_{k+1} g_{k+1} - f_k g_k$
$= f_{k+1} g_{k+1} - f_{k+1} g_k + f_{k+1} g_k - f_k g_k$
$= f_{k+1}(g_{k+1} - g_k) + (f_{k+1} - f_k) g_k$

$= f_{k+1}\Delta g_k + g_k \Delta f_k$

$= R.H.S$ (proved).

(viii) $\Delta\left(\frac{f_k}{g_k}\right) = \frac{g_k \Delta f_k - f_k \Delta g_k}{g_k\, g_{k+1}}$

L.H.S $\Delta\left(\frac{f_k}{g_k}\right) = \frac{f_{k+1}}{g_{k+1}} - \frac{f_k}{g_k}$

$= \frac{f_{k+1} g_k - f_k g_{k+1}}{g_k \cdot g_{k+1}}$

$= \frac{f_{k+1} g_k - f_k g_k + f_k g_k - g_{k+1} f_k}{g_{k+1} \cdot g_k}$

$= \frac{g_k (f_{k+1} - f_k) - f_k (g_{k+1} - g_k)}{g_{k+1} \cdot g_k}$

$= \frac{g_k \Delta f_k - f_k \Delta g_k}{g_{k+1} \cdot g_k} = R.H.S$

(proved).

(ix) $hD = \log(1+\Delta) = -\log(1-\nabla)$

As we know that $e^{hD} = E$

$\Rightarrow hD = \log(E)$

$hD = \log(1+\Delta) \longrightarrow (i)$

Again $e^{hD} = E$

$\Rightarrow hD = \log(E)$

$hD = \log(1-\nabla)^{-1}$

$hD = -\log(1-\nabla) \longrightarrow (ii)$

From (i) and (ii)

$hD = \log(1+\Delta) = -\log(1-\nabla)$

(proved).

(x) $\Delta^k f_i = \sum_{r=0}^{k} (-1)^{k+r} \binom{k}{r} f_{i+r}$

L.H.S $= \Delta^k f_i = (E-1)^k f_i$

$= (-1)^k (1-E)^k f_i$

$= (-1)^k \left\{ \sum_{r=0}^{k} \binom{k}{r} (1)^{k-r} (-E)^r \right\} f_i$

$\because$ by binomial Theorem.

$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$

$= \left\{ \sum_{r=0}^{k} (-1)^k \binom{k}{r} (-1)^r E^r \right\} f_i$

$= \sum_{r=0}^{k} (-1)^{k+r} \binom{k}{r} f_{i+r}$

$\because E^n f(x) = f(x+nh)$

$= R.H.S$ $\quad \therefore E^r f_i = f_{i+r}$

(xi) $\Delta^r y_k = \nabla^r y_{k+r}$

L.H.S $= \Delta^r y_k$

$= (E-1)^r y_k$

$= \left(\sum_{i=0}^{r} \binom{r}{i} E^{r-i} (-1)^i \right) y_k$

$= \left(\sum_{i=0}^{r} (-1)^i \binom{r}{i} E^{r-i} \right) y_k$

$= \sum_{i=0}^{r} (-1)^i \binom{r}{i} y_{k+r-i}$

R.H.S $= \nabla^r y_{k+r}$

$= (1-E^{-1})^r y_{k+r}$

$= \left(\sum_{i=0}^{r} \binom{r}{i} (1)^{r-i} (-E)^{-i} \right) y_{k+r}$

$= \left(\sum_{i=0}^{r} (-1)^i \binom{r}{i} E^{-i} \right) y_{k+r}$

$= \sum_{i=0}^{r} (-1)^i \binom{r}{i} y_{k+r-i}$

L.H.S $=$ R.H.S (proved).

(xii) $e^x = \frac{\Delta^2}{E} e^x \cdot \frac{Ee^x}{\Delta^2 e^x}$

R.H.S $\frac{\Delta^2 e^x}{E} \cdot \frac{Ee^x}{\Delta^2 e^x}$

$= \Delta^2 E^{-1} e^x \cdot Ee^x/\Delta^2 e^x$

$= \Delta^2 e^{x-h} \cdot Ee^x/\Delta^2 e^x$

$= e^{-h} \cancel{\Delta^2} e^x \cdot \frac{Ee^x}{\cancel{\Delta^2} e^x}$

$= e^{-h} E e^x$

$= e^{-h} e^{x+h} \quad (\because Ef(x) = f(x+h))$

$= e^x = L.H.S$ (proved)

(xiii) $\mu\delta = \frac{1}{2}\Delta E^{-1} + \frac{1}{2}\Delta = \frac{1}{2}(\Delta + \nabla)$.

$\mu\delta = (E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2})$

$= \frac{E - E^{-1}}{2} \longrightarrow (i) \quad \because (x-a)(x+a) = x^2 - a^2$

$\frac{1}{2}\Delta E^{-1} + \frac{1}{2}\Delta = \frac{\Delta E^{-1} + \Delta}{2}$

$= \frac{-(1-E)E^{-1} + (E-1)}{2}$

$= \frac{1 - E^{-1} + E - 1}{2} = \frac{E - E^{-1}}{2} \longrightarrow (ii)$

$\frac{1}{2}(\Delta + \nabla) = \frac{1}{2}(E - \cancel{1} + \cancel{1} - E^{-1})$

$= \frac{E - E^{-1}}{2} \longrightarrow (iii)$

From (i), (ii) and (iii)

$\mu\delta = \frac{1}{2}\Delta E^{-1} + \frac{1}{2}\Delta = \frac{1}{2}(\Delta + \nabla)$ (proved).

(xiv) $\Delta - \frac{1}{2}\delta^2 = \delta\sqrt{1 + \delta^2/4}$

L.H.S $= \Delta - \frac{1}{2}\delta^2$

$= E - 1 - \frac{1}{2}(E^{1/2} - E^{-1/2})^2$

$= E - 1 - \frac{1}{2}(E + E^{-1} - 2)$

$= E - 1 - \frac{E}{2} - \frac{E^{-1}}{2} + 1$

$= \frac{E - E^{-1}}{2}$

R.H.S $= \delta\sqrt{1 + \frac{\delta^2}{4}}$

$= (E^{1/2} - E^{-1/2})\sqrt{1 + \frac{(E^{1/2} - E^{-1/2})^2}{4}}$

$= (E^{1/2} - E^{-1/2})\sqrt{\frac{4 + E + E^{-1} - 2}{4}}$

$= (E^{1/2} - E^{-1/2})\sqrt{\frac{E + E^{-1} + 2}{4}}$

$= E^{1/2} - E^{-1/2}\sqrt{\frac{(E^{1/2} + E^{-1/2})^2}{(2)^2}}$

$= (E^{1/2} - E^{-1/2})\left(\frac{E^{1/2} + E^{-1/2}}{2}\right)$

$= \frac{E - E^{-1}}{2}$

L.H.S = R.H.S (proved).

(ii) $e^x = \left(\frac{\Delta^2}{E}\right) e^x \cdot \frac{E e^x}{\Delta^2 e^x}$

R.H.S $\frac{\Delta^2 e^x}{E} \cdot \frac{E e^x}{\Delta^2 e^x}$

$= \Delta^2 E^{-1} e^x \cdot E e^x / \Delta^2 e^x$

$= \Delta^2 e^{x-h} \cdot E e^x / \Delta^2 e^x$

$= e^{-h} \Delta^2 e^x \cdot \frac{E e^x}{\Delta^2 e^x}$

$= e^{-h} E e^x$

$= e^{-h} e^{x+h} \qquad (\because E f(x) = f(x+h))$

$= e^x = L.H.S \quad (\text{proved})$

(iii) $\mu\delta = \frac{1}{2}\Delta E^{-1} + \frac{1}{2}\Delta = \frac{1}{2}(\Delta + \nabla)$

$\mu\delta = \frac{(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2})}{2}$

$= \frac{E - E^{-1}}{2} \longrightarrow (i) \qquad \because (x-a)(x+a) = x^2 - a^2$

$\frac{1}{2}\Delta E^{-1} + \frac{1}{2}\Delta = \frac{\Delta E^{-1} + \Delta}{2}$

$= \frac{(E-1)E^{-1} + (E-1)}{2}$

$= \frac{1 - E^{-1} + E - 1}{2} = \frac{E - E^{-1}}{2} \longrightarrow (ii)$

$\frac{1}{2}(\Delta + \nabla) = \frac{1}{2}(E - 1 + 1 - E^{-1})$

$= \frac{E - E^{-1}}{2} \longrightarrow (iii)$

From (i), (ii) and (iii)

$\mu\delta = \frac{1}{2}\Delta E^{-1} + \frac{1}{2}\Delta = \frac{1}{2}(\Delta + \nabla) \quad (\text{proved})$

(iv) $\Delta - \frac{1}{2}\delta^2 = \delta\sqrt{1 + \delta^2/4}$

L.H.S $= \Delta - \frac{1}{2}\delta^2$

$= E - 1 - \frac{1}{2}(E^{1/2} - E^{-1/2})^2$

$= E - 1 - \frac{1}{2}(E + E^{-1} - 2)$

$= E - 1 - \frac{E}{2} - \frac{E^{-1}}{2} + 1$

$= \frac{E - E^{-1}}{2}$

R.H.S $= \delta\sqrt{1 + \frac{\delta^2}{4}}$

$= (E^{1/2} - E^{-1/2})\sqrt{1 + \frac{(E^{1/2} - E^{-1/2})^2}{4}}$

$= (E^{1/2} - E^{-1/2})\sqrt{\frac{4 + E + E^{-1} - 2}{4}}$

$= (E^{1/2} - E^{-1/2})\sqrt{\frac{E + E^{-1} + 2}{4}}$

$= E^{1/2} - E^{-1/2}\sqrt{\frac{(E^{1/2} + E^{-1/2})^2}{(2)^2}}$

$= (E^{1/2} - E^{-1/2})\left(\frac{E^{1/2} + E^{-1/2}}{2}\right)$

$= \frac{E - E^{-1}}{2}$

L.H.S = R.H.S (proved).

(xv) $hD = \sinh^{-1}(\mu\delta)$

Rewrite problem we have

$\sinh(hD) = \mu\delta$

$= \left(\frac{E^{1/2}+E^{-1/2}}{2}\right)(E^{1/2}-E^{-1/2})$

$= \frac{E-E^{-1}}{2} \quad \because E = e^{hD}$

$= \frac{e^{hD}-e^{-hD}}{2}$

$= \sinh(hD)$ proved

(xvi) $\mu = (1-\frac{\nabla}{2})(1-\nabla)^{-1/2}$

R.H.S

$(1-\frac{\nabla}{2})(1-\nabla)^{-1/2}$

$= \frac{2-(1-E^{-1})}{2} \cdot \{1-(1-E^{-1})\}^{-1/2}$

$= \frac{2-1+E^{-1}}{2}(1-1+E^{-1})^{-1/2}$

$= \frac{1+E^{-1}}{2}(E^{1/2}) = \frac{E^{1/2}+E^{-1/2}}{2} = \mu$

$=$ R.H.S

(proved).

(xvii) $(E^{1/2}+E^{-1/2})(1+\Delta)^{1/2} = 2+\Delta$

L.H.S

$= (E^{1/2}+E^{-1/2})(1+E-1)^{1/2}$

$= (E^{1/2}+E^{-1/2})(E)^{1/2}$

$= E+E^{-1/2+1/2}$

$= 1+\Delta+E^{0}$

$= 1+\Delta+1$

$= 2+\Delta$

$=$ R.H.S (proved)

(xviii) $\mu(f_k g_k) = \mu f_k \mu g_k + \frac{1}{4}\delta f_k \delta g_k$.

L.H.S $\mu(f_k g_k)$

$= \frac{E^{1/2}+E^{-1/2}}{2}(f_k g_k)$

$= \frac{1}{2}\{E^{1/2}(f_k g_k) + E^{-1/2}(f_k g_k)\}$

$= \frac{1}{2} f_{k+\frac{1}{2}} g_{k+\frac{1}{2}} + \frac{1}{2} f_{k-\frac{1}{2}} g_{k-\frac{1}{2}}$

R.H.S $= \mu f_k \mu g_k + \frac{1}{4}\delta(f_k g_k)$

$= \left(\frac{E^{1/2}+E^{-1/2}}{2}\right)(f_k) \cdot \left(\frac{E^{1/2}+E^{-1/2}}{2}\right) g_k + \frac{1}{2}(E^{1/2}-E^{-1/2}) f_k \delta g_k$

$= \left(\frac{1}{2}E^{1/2}f_k + \frac{1}{2}E^{-1/2}f_k\right)\left(\frac{1}{2}E^{1/2}g_k + \frac{E^{-1/2}g_k}{2}\right) +$

$\left(\frac{1}{2}E^{1/2}f_k - \frac{1}{2}E^{-1/2}f_k\right)\left(\frac{1}{2}E^{1/2}g_k - \frac{1}{2}E^{-1/2}g_k\right)$

$= \left(\frac{1}{2}f_{k+\frac{1}{2}} + \frac{1}{2}f_{k-\frac{1}{2}}\right)\left(\frac{1}{2}g_{k+\frac{1}{2}} + \frac{1}{2}g_{k-\frac{1}{2}}\right) +$

$\left(\frac{1}{2}f_{k+\frac{1}{2}} - \frac{1}{2}f_{k-\frac{1}{2}}\right)\left(\frac{1}{2}g_{k+\frac{1}{2}} - \frac{1}{2}g_{k-\frac{1}{2}}\right)$

$= \frac{1}{4} f_{k+\frac{1}{2}} g_{k+\frac{1}{2}} + \cancel{\frac{1}{4} f_{k+\frac{1}{2}} g_{k-\frac{1}{2}}} + \cancel{\frac{1}{4} f_{k-\frac{1}{2}} g_{k+\frac{1}{2}}} +$

$\frac{1}{4}(f_{k-\frac{1}{2}} g_{k-\frac{1}{2}}) + \frac{1}{4}(f_{k+\frac{1}{2}} g_{k+\frac{1}{2}}) - \cancel{\frac{1}{4}(f_{k+\frac{1}{2}} g_{k-\frac{1}{2}})}$

$- \cancel{\frac{1}{4}(f_{k-\frac{1}{2}} g_{k+\frac{1}{2}})} + \frac{1}{4}(f_{k-\frac{1}{2}} g_{k-\frac{1}{2}})$

$= \frac{1}{2} f_{k+\frac{1}{2}} g_{k+\frac{1}{2}} + \frac{1}{2} f_{k-\frac{1}{2}} g_{k-\frac{1}{2}}$

L.H.S = R.H.S (proved).

EXERCISE

Show that :-

(i) $\Delta \log x_k = \log\left(1 + h/x_k\right)$

(ii) $\nabla \log x_k = -\log\left(1 - h/x_k\right)$

(iii) $\Delta \sqrt{f_k} = \dfrac{\Delta f_k}{\sqrt{f_k} + \sqrt{f_{k+1}}}$

(iv) $(1 + \Delta)(1 - \nabla) = 1$

(v) $\Delta + \nabla = \dfrac{\Delta}{\nabla} - \dfrac{\nabla}{\Delta}$

(vi) $\Delta\left(\dfrac{1}{f(x)}\right) = -\dfrac{\Delta f(x)}{f(x)\,f(x+1)}$

(vii) $\dfrac{\Delta}{E} \sin x = 2\cos\left(x - \dfrac{h}{2}\right)\sin\left(\dfrac{h}{2}\right)$

(viii) $\Delta = \delta E^{1/2} = E^{1/2}\delta$

(ix) $\mu^2 = 1 + \dfrac{\delta^2}{4}$

(x) $\mu\left(\dfrac{f_k}{g_k}\right) = \dfrac{\mu(f_k)\mu(g_k) - \frac{1}{4}\delta(f_k)\delta(g_k)}{g_{k-\frac{1}{2}} \cdot g_{k+\frac{1}{2}}}$

(xi) $\Delta(1 + \Delta)^{-1/2} = \nabla(1 - \nabla)^{-1/2} = \delta$

INTERPOLATION

* **Definition :-** It is the technique of estimating the value of a function $y = f(x)$ for any intermediate value of the independent variable 'x' when the value of function corresponding to a number of values of x are given.

Example:

If given for function $y = f(x)$

x	0	1	2	3	4
$f(x)$	3	7	5.6	11	15

Then to find $f(x)$; $0 < x < 4$ is called interpolation.

* **Extrapolation :-** The process of estimating the value of function $y = f(x)$ outside the range of given values of variable is called extrapolation.

Example: From above example to calculate $f(-0.05)$ or $f(4.5)$ etc is called extrapolation.

* **Arguments and Entries:**

If for a function $y = f(x)$ for value of independent variable x say $x_0, x_1, \ldots, x_n$. It is given $f(x_0), f(x_1), \ldots, f(x_n)$, then $x_0, x_1, \ldots, x_n$ are called arguments while $f(x_0), f(x_1), \ldots, f(x_n)$ are called entries.

* **Equally Spaced Arguments:**

If $x_0, x_1, \ldots, x_n$ are arguments for the variable x s.t $x_k - x_{k-1} = h$ $\forall$ $k = 1, 2, \ldots n$, then these arguments are said to be equally spaced arguments. where h is called step size or interval difference.

* **Unequally Spaced Arguments:**

If the arguments are not equally spaced, then they called unequally spaced.

* **Interpolating Polynomial:-**

Suppose we are given the values of function $f(x)$ at certain points $x_0, x_1, x_2, \ldots, x_n$ and we want to estimate $f(\alpha)$ for some $\alpha \in [x_0, x_n]$ Then we assume that function can be approximated by a polynomial. This polynomial is called interpolating polynomial.

* **Forward difference formula:-**

we know that

$$\Delta f(x) = f(x+h) - f(x)$$

then we note that

$$\begin{aligned}\Delta^2 f(x) &= \Delta(\Delta f(x)) \\ &= \Delta(f(x+h) - f(x)) \\ &= f(x+2h) - f(x+h) - f(x+h) + f(x) \\ &= f(x+2h) - 2f(x+h) + f(x).\end{aligned}$$

If we denote $f(x_0)$ by y_0, $f(x_1)$ by y_1, ..., $f(x_n)$ by y_n then

$$\begin{aligned}\Delta y_0 &= y_1 - y_0 \\ \Delta^2 y_0 &= \Delta(\Delta y_0) \\ &= \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0 \\ \Delta^3 y_0 &= \Delta(\Delta^2 y_0) = \Delta(\Delta y_1 - \Delta y_0) \\ &= \Delta^2 y_1 - \Delta^2 y_0\end{aligned}$$

in general

$$\begin{aligned}\Delta y_r &= y_{r+1} - y_r \\ \Delta^2 y_r &= \Delta y_{r+1} - \Delta y_r \\ \Delta^3 y_r &= \Delta^2 y_{r+1} - \Delta^2 y_r \\ &\vdots \\ \Delta^n y_r &= \Delta^{n-1} y_{r+1} - \Delta^{n-1} y_r.\end{aligned}$$

Forward difference Table :-

x	y	Δ	Δ^2	Δ^3	Δ^4
x_0	y_0				
		$\Delta y_0 = y_1 - y_0$			
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$		
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$	
x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$
		$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$	
x_3	y_3		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		
		$\Delta y_3 = y_4 - y_3$			
x_4	y_4				

Interpolating Polynomial

Types

- Equally Spaced Data
  - (Methods)
  - (i) Newton's Formulae (Forward & Backward).
  - (ii) Gauss Formulae (Forward & Backward)
  - (iii) Stirling's Formula
  - (iv) Bessel's Formula
  - (v) Everett's Formula
- Unequally Spaced Data
  - (Methods)
  - (i) Aitken's Formula
  - (ii) Lagranges's Formula
  - (iii) Newton's divided difference Formula.

Remark:-

(i) We use formulae of unequally spaced data to calculate equally spaced data interpolating polynomial but equally spaced formulae only applicable on equally spaced data.

(ii) Newton's Difference Formulae is also called Gregory newton's difference formulae.

For Equally Spaced Data

(i) Newton's Forward dif. Formula:-

If $x_0, x_1, \ldots, x_n$ are equally spaced, then for any x,

$$f(x) = f(x_0) + u\,\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0) + \ldots\ldots$$

where $x_0 = I^{st}$ argument

$u = \dfrac{x - x_0}{h}$, h is stepsize

Proof:-

Given $u = \dfrac{x - x_0}{h}$

$\Rightarrow uh = x - x_0$

$\Rightarrow x = x_0 + uh$

$f(x) = f(x_0 + uh)$

$= E^u f(x_0) \quad \because Ef(x) = f(x+h)$

$= (1+\Delta)^u f(x_0) \quad \because E = 1 + \Delta$

$$f(x) = \left[1 + u\Delta + \frac{u(u-1)}{2!}\Delta^2 + \frac{u(u-1)(u-2)}{3!}\Delta^3 + \ldots\right] f(x_0)$$

$$f(x) = f(x_0) + u\,\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0) + \ldots\ldots$$

(proved)

Note:

This formula is used when estimated value lies in beginning of given data.

Question:

$\sin 45° = 0.707$, $\sin 50° = 0.7760$, $\sin 55° = 0.8192$, $\sin 60° = 0.8660$, Find $\sin 52°$ by Newton's forward dif. formula.

Solution:-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	0.707			
		0.0589		
50	0.7760		-0.0057	
		0.0532		-0.0007
55	0.8192		-0.0064	
		0.0468		
60	0.8660			

$x_0 = 45$, $x = 52$, $h = 5$

$$u = \frac{x - x_0}{h} = \frac{52 - 45}{5} = \frac{7}{5} = 1.4$$

By Newton's Forward difference Formula

$$f(x) = f(x_0) + u\,\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0) + \ldots$$

$$f(52) = 0.707 + (1.4)(0.0589) + \frac{(1.4)(0.4)}{2!}(-0.0057) + \frac{(1.4)(0.4)(-0.6)}{3!}(-0.0007)$$

$f(52) = 0.7880$

(Answer)

Question:-

$\sqrt[3]{29} = 3.0727$, $\sqrt[3]{30} = 3.1074$

$\sqrt[3]{27} = 3.000$, $\sqrt[3]{28} = 3.0367$ Find $\sqrt[3]{26}$ using N.F.D. Formula

Solution:- As $f(x) = \sqrt[3]{x}$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
27	3.00			
		0.0367		
28	3.0367		−0.0011	
		0.0358		0
29	3.0727		−0.0011	
		0.0347		
30	3.1074			

As $x = 26$, $x_0 = 27$, $h = 1$

$$u = \frac{x - x_0}{h} = \frac{26-27}{1} = -1$$

By Newton's forward difference formula:

$$f(x) = f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0) + \ldots\ldots$$

$$f(26) = 3.00 + (-1)(0.0367) + \frac{(-1)(-2)}{2!}(-0.0011) + \frac{(-1)(-2)(-3)}{3!}(0)$$

$\Rightarrow \sqrt[3]{26} = 3.00 - 0.0367 - 0.0011$
$= 2.9620$

(Ans)

Question:-

Find polynomial which satisfies following data

x	1	2	3	4
$f(x)$	1	−1	1	−1

Solution:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	$\boxed{1}$			
		$\boxed{-2}$		
2	−1		$\boxed{4}$	
		2		$\boxed{8}$
3	1		−4	
		−2		
4	−1			

$x_0 = 1$

x is a Variable

$h = 1$

$$u = \frac{x - x_0}{h} = \frac{x-1}{1} = x - 1$$

By Newton's Forward difference Formula:

$$f(x) = f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0) + \ldots\ldots$$

$$f(x) = 1 - (x-1)(-2) + \frac{(x-1)(x-2)(4)}{2!} - \frac{(x-1)(x-2)(x-3)(-8)}{3!}$$

$$= 1 - 2x + 2 + 2(x^2 - 2x - x + 2) - \frac{4}{3}(x^3 - 6x^2 + 11x - 6)$$

$$= 1 - 2x + 2 + 2x^2 - 4x - 2x + 8 - \frac{4}{3}x^3 + 8x^2 - \frac{44}{3}x - 8$$

$$= -\frac{4}{3}x^3 + 10x^2 - \frac{68}{3}x + 15$$

$$= \frac{1}{3}(-4x^3 + 30x^2 - 68x + 45)$$

(Answer)

Newton's Backward Difference Formula:

If for a function f, its value at $x_0, x_1, \ldots, x_n$ are $f(x_0), f(x_1), \ldots, f(x_n)$, then for any value of x

$$f(x) = f(x_n) + u\nabla f(x_n) + \frac{u(u+1)}{2!}\nabla^2 f(x_n) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(x_n) + \cdots\cdots$$

Proof:- Now $u = \frac{x - x_n}{h}$

where x_n is last argument and h is interval difference.

$$\Rightarrow x = x_n + uh$$

$$f(x) = f(x_n + uh)$$

$$= E^{+u} f(x_n)$$

$$= [(1-\nabla)^{-1}]^u f(x_n)$$

$$= (1-\nabla)^{-u} f(x_n). \qquad \because E = (1-\nabla)^{-1}$$

$$= \left[1 + u\nabla + \frac{u(u+1)}{2!}\nabla^2 + \frac{u(u+1)(u+2)}{3!}\nabla^3 + \cdots\cdots\right] f(x_n)$$

$$\Rightarrow f(x) = f(x_n) + u\nabla f(x_n) + \frac{u(u+1)}{2!}\nabla^2 f(x_n)$$

$$+ \frac{u(u+1)(u+2)}{3!}\nabla^3 f(x_n) + \cdots\cdots$$

Note:-

This formula is used when estimating value lies at the end of the given data.

Question:

The population of town is given below. Estimate population for year 1965.

Year	1931	1941	1951	1961	1971
Popul.	46	66	81	93	101

Solution:

x	y	1st Diff.	2nd Diff.	3rd Diff.	4th Diff.
1931	46				
		20			
1941	66		-5		
		15		2	
1951	81		-3		$\boxed{-3}$
		12		$\boxed{-1}$	
1961	93		$\boxed{-4}$		
		$\boxed{8}$			
1971	$\boxed{101}$				

$x_n = 1971$

$x = 1965$

$h = 10$

$u = \frac{x - x_n}{h} = \frac{1965 - 1971}{10} = \frac{-6}{10} = -0.6$

By Newton's Backward diff. formula

$f(x) = f(x_n) + u\nabla f(x_n) + \frac{u(u+1)}{2!}\nabla^2 f(x_n) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(x_n) + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 f(x_n) + \ldots\ldots$

$f(1965) = 101 + (-0.6)(8) + \frac{(-0.6)(0.4)}{2!}(-4) + \frac{(-0.6)(-0.6+1)(-0.6+2)}{3!}(-1) + \frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{4!}(-3)$

$= 101 - 4.8 + 0.48 + 0.0560 + 0.1008$

$= 96.837$

(Answer)

Question:-

By Constructing difference table, find 7th term as well as general term of Sequence 0, 0, 2, 6, 12, 20...

Solution:

It is easy to calculate general term by using Newton's forward or Newton's backward difference formula.

The given difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	$\boxed{0}$			
		$\boxed{0}$		
2	0		$\boxed{2}$	
		2		$\boxed{0}$
3	2		2	
		4		0
4	6		2	
		6		$\boxed{0}$
5	12		$\boxed{2}$	
		$\boxed{8}$		
6	$\boxed{20}$			

By Newton's forward difference table

$$f(x) = f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0) + \cdots\cdots$$

$$f(x) = 0 + (x-1)(0) + \frac{(x-2)(x-1)(2)}{2} + 0$$

$$= 0 + 0 + (x-2)(x-1)$$

$$= x^2 - 3x + 2$$

which is general term of sequence.

$\because u = \frac{x - x_0}{h}$

$u = \frac{x-1}{1}$

To find 7^{th} term put $x = 7$

$$f(x) = x^2 - 3x + 2$$

$$f(7) = 49 - 21 + 2 = 30.$$

By Newton's Backward Diff. Formula :-

$x_n = 6$

$h = 1$

$u = \frac{x - x_n}{1} = x - 6$

$$f(x) = f(x_n) + u\nabla f(x_n) + \frac{u(u+1)}{2!}\nabla^2 f(x_n) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(x_n) + \cdots\cdots$$

$$= 20 + (x-6)(8) + \frac{(x-6)(x-5)(2)}{2!} + \frac{(x-6)(x-5)(x-4)}{3!}(0)$$

$$= 20 + 8x - 48 + x^2 - 11x + 30$$

$$= x^2 - 3x + 2$$

Which is general term of sequence.

For 7^{th} term

put $x = 7$

$$f(7) = (7)^2 - 3(7) + 2$$

$$= 49 - 21 + 2$$

$$= 30.$$

(Answer)

Question :-

Find a polynomial which satisfies $y = |x|$ at $x = -2, -1, 0, 1, 2$ by Newton's backward diff. formula.

Solution:-

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
-2	2				
		-1			
-1	1		0		
		-1			
0	0		2	2	
		1			4
1	1		0	-2	
		1			
2	2				

$h = 1 \quad , \quad x_n = 2$

$u = \dfrac{x - x_n}{h} = x - 2$

By Newton's backward diff. Formula

$$f(x) = f(x_n) + u\nabla f(x_n) + \frac{u(u+1)}{2!}\nabla^2 f(x_n) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(x_n)$$

$$+ \cdots\cdots$$

$$= 2 + (x-2)(1) + \frac{(x-2)(x-1)}{2!}(0)$$

$$+ \frac{(x-2)(x-1)(x)}{3!}(-2) +$$

$$\frac{(x-2)(x-1)(x)(x+1)}{4!}(-4)$$

$$= 2 + x - 2 - \frac{1}{3}x^3 + x^2 - \frac{2}{3}x - \frac{1}{6}x^4 + \frac{1}{3}x^3$$

$$+ \tfrac{1}{6}x^2 - \tfrac{1}{3}x$$

$$f(x) = \frac{-1}{6}x^4 + \frac{7}{6}x^2 \quad \text{(Required polynomial)}$$

* Centeral DIFFERENCE Table:-

Centeral difference is defined as

$$\delta(f(x)) = f(x + h/2) - f(x - h/2)$$

or

$$\delta f(x) = f(x+h) - f(x-h)$$

In the case when stepsize is $2h$ instead of h.

Centeral Difference Table is

x	$f(x)$	1st Diff.	2nd Diff.	3rd Diff.	4th Diff.
x_{-2}	y_{-2}				
		Δy_{-2}			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		
		Δy_{-1}		$\Delta^3 y_{-2}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$
		Δy_0		$\Delta^3 y_{-1}$	
x_1	y_1		$\Delta^2 y_0$		
		Δy_1			
x_2	y_2				

THEORHM:

The n^{th} forward (backward) difference of a polynomial of degree n is constant and all higher order differences are zero.

Proof:- Let $f(x)$ be a polynomial of degree n

i.e $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n$

Then

I^{st} order forward difference is

$$\Delta f(x) = f(x+h) - f(x) \longrightarrow (i)$$

As

$$f(x+h) = a_0 + a_1(x+h) + a_2(x+h)^2 + a_3(x+h)^3 + \dots + a_n(x+h)^n$$

$$= a_0 + a_1(x+h) + a_2(x^2+2xh+h^2) + a_3(x^3+3x^2h+3xh^2+h^3) + \dots + a_n\left(x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \dots + h^n\right)$$

Put $f(x+h)$ and $f(x)$ in (i) we have

$$\Delta f(x) = \cancel{a_0} + a_1(x+h) + a_2(x+h)^2 + a_3(x+h)^3 + \dots + a_n(x+h)^n - \cancel{a_0} - a_1 x - a_2 x^2 - a_3 x^3 - \dots - a_n x^n$$

$$\Delta f(x) = a_1(x+h-x) + a_2[(x+h)^2 - x^2] + a_3[(x+h)^3 - x^3] + \dots + a_n[(x+h)^n - x^n]$$

$$\Delta f(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \dots + b_{n-1} x^{n-1}$$

Now

$$\Delta^2 f(x) = \Delta(\Delta f(x))$$

$$= \Delta(b_0 + b_1 x + b_2 x^2 + \dots + b_{n-1} x^{n-1})$$

$$= c_0 + c_1 x + c_2 x^2 + \dots + c_{n-2} x^{n-2}$$

Hence

I^{st}	Forward difference of	$f(x)$ =	polynomial of degree	$n-1$
2^{nd}	" "	$f(x)$ =	" " "	$n-2$
3^{rd}	" "	$f(x)$ =	" " "	$n-3$
⋮	⋮	⋮	⋮	⋮
n^{th}	" "	$f(x)$ =	" " "	$n-n=0$

$\Rightarrow$ n^{th} forward difference of $f(x)$ is a polynomial of degree zero which is constant.

$$\Delta^n f(x) = C$$

$$\Delta^{n+1} f(x) = \Delta(\Delta^n f(x))$$

$$= \Delta(C)$$

$$= C - C = 0$$

$$\Delta^{n+2} f(x) = \Delta(\Delta^{n+1} f(x))$$

$$= \Delta(0)$$

$$= 0 - 0 = 0$$

Hence all forward differences of order greater then n are zero.

Similarly: It is easy to prove for backward difference.

Hint: $\nabla f(x) = f(x) - f(x-h)$.

* (II) GAUSS CENTERAL DIFFERENCE FORMULAE:-

(i) Gauss foward formula:-

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{u(u-1)(u+1)(u-2)}{4!}\Delta^4 y_{-2} + \cdots\cdots$$

where $u = \frac{x - x_0}{h}$.

Proof:-

By Newton's Forward difference formula is

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$
$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \cdots\cdots \longrightarrow (i)$$

Now as
$$\Delta^{n+1} y_{i-1} = \Delta(\Delta^n y_{i-1})$$
$$= \Delta^n (y_{i-1+1} - y_{i-1})$$
$$= \Delta^n (y_i - y_{i-1})$$
$$\Delta^{n+1} y_{i-1} = \Delta^n y_i - \Delta^n y_{i-1}$$
$$\Delta^n y_i = \Delta^n y_{i-1} + \Delta^{n+1} y_{i-1}$$

From above
$$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$$
and
$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$$
$$= \Delta^3 y_{-1} + \Delta^4 y_{-2} + \Delta^5 y_{-2}$$

Using all these values in (i) we have

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\left[\Delta^2 y_{-1} + \Delta^3 y_{-1}\right] +$$
$$\frac{u(u-1)(u-2)}{3!}\left[\Delta^3 y_{-1} + \Delta^4 y_{-2} + \Delta^5 y_{-2}\right] + \cdots\cdots$$
$$= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \left[\frac{u(u-1)}{2!} + \frac{u(u-1)(u-2)}{3!}\right]\Delta^3 y_{-1}$$
$$+ \cdots\cdots$$
$$= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1}$$
$$+ \cdots\cdots$$

which is required Gauss forward formula for equal interval.

Note:-

Gauss forward formula is used when no. of entries below the line (centeral line) is greater then above centeral line.

* **Question:-** Find value of y corresponding to $x=5$ from table

x : 2 : 4 : 6 : 8 : 10

y : 0 : 0 : 1 : 0 : 0

by Gauss forward formula.

Solution:-

The difference table is given by

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_{-2}	2	0				
			0			
x_{-1}	4	0		1		
	→ 5		1		-3	
x_0	6	1		-2		6
			-1		3	
x_1	8	0		1		
			0			
x_2	10	0				

centeral Line

put $u = \frac{x - x_0}{h} = \frac{x-6}{2}$

at $x=5$, $u = \frac{5-6}{2} = -0.5$

By Gauss Forward Formula:-

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{u(u-1)(u+1)(u-2)}{4!}\Delta^4 y_{-2}$$

$$f(5) = 1 + (-0.5)(-1) + \frac{(-0.5)(-1.5)}{2}(-2) +$$

$$+ \frac{(-0.5)(-1.5)(0.5)}{3!}(3) + \frac{(-0.5)(-1.5)(0.5)(-2.5)}{4!}(6)$$

$$f(5) = 1 + 0.5 - 0.75 + 0.19 - 0.23$$

$$= 0.71$$

(Answer).

* **Question:**

Find value of y corresponding to $x = 4.7$ From table by Gauss Forward Formula.

x : 3 : 4 : 5 : 6

y : 6 : 24 : 60 : 120

Solution:-

The difference table is given by.

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_{-1}	3	6			
			18		
x_0	4	24		18	
	4.7		36		6
x_1	5	60		24	
			60		
x_2	6	120			

centeral line

$u = \frac{x - x_0}{h} = \frac{x-4}{1}$

At $x=4.7$ $u = 4.7 - 4 = 0.7$

By Gauss Forward Formula:

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1}$$

$$+ \dots\dots$$

$$f(4.7) = 24 + (0.7)(36) + \frac{(0.7)(-0.3)}{2!}(18)$$

$$+ \frac{(0.7)(-0.3)(1.7)}{3!}(6)$$

$$= 46.95$$

(Answer)

(ii) Gauss Backward Formula :-

$$f(x) = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{u(u+1)(u-1)}{3!}\Delta^3 y_{-2} + \frac{u(u+1)(u-1)(u+2)}{4!}\Delta^4 y_{-2} + \cdots\cdots$$

where $u = \frac{x - x_0}{h}$.

Proof:- By Newton's Forward difference formula

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \cdots\cdots$$

As
$$\Delta^{n+1} y_{i-1} = \Delta(\Delta^n y_{i-1})$$
$$= \Delta^n(\Delta y_{i-1})$$
$$= \Delta^n(y_{i-1+1} - y_{i-1})$$
$$= \Delta^n y_i - \Delta^n y_{i-1}$$
$$\Delta^n y_i = \Delta^n y_{i-1} + \Delta^{n+1} y_{i-1}$$
$$\Rightarrow \Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$$
and
$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$$
$$= \Delta^3 y_{-1} + \Delta^4 y_{-2} + \Delta^5 y_{-2}$$

using all these value (also put Δy_0's value).

$$f(x) = y_0 + u[\Delta y_{-1} + \Delta^2 y_{-1}] + \frac{u(u-1)}{2!}[\Delta^2 y_{-1} + \Delta^3 y_{-2} + \Delta^4 y_{-2}]$$
$$+ \frac{u(u-1)(u-2)}{3!}(\Delta^3 y_{-2} + 2\Delta^4 y_{-2} + \Delta^5 y_{-3} + \Delta^6 y_{-3})$$
$$+ \cdots\cdots$$

$$= y_0 + u\Delta y_{-1} + \left[u + \frac{u(u-1)}{2!}\right]\Delta^2 y_{-1} + \left[\frac{u(u-1)}{2!} + \frac{u(u-1)(u-2)}{3!}\right]\Delta^3 y_{-2} + \cdots$$

$$= y_0 + u\Delta y_{-1} + u\left[1 + \frac{u-1}{2!}\right]\Delta^2 y_{-1} + \frac{u(u-1)}{2!}\left[1 + \frac{u-2}{3}\right]\Delta^3 y_{-2} + \cdots\cdots$$

$$= y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{u(u+1)(u-1)}{3!}\Delta^3 y_{-2} + \cdots\cdots$$

is required formula.

Question:- Find the population of town for 1974 of data by gauss backward Formula.

Year: 1939 : 1949 : 1959 : 1969 : 1979 : 1989

Popu: 12 : 15 : 20 : 27 : 39 : 52.

Solution:-

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
1939	12					
		3				
1949	15		2			
		5		0		
1959	20		2		3	
		7		3		-10
x_0 1969	27		5		-7	
→ 1974		12		-4		
1979	39		1			
		13				
1989	52					

Taking $x_0 = 1969$, $x = 1974$, $h = 10$

$$u = \frac{x - x_0}{h} = \frac{1974 - 1969}{10} = 0.5$$

By Gauss backward Formula

$$f(x) = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{u(u+1)(u-1)}{3!}\Delta^3 y_{-2} + \frac{u(u+1)(u-1)(u+2)}{4!}\Delta^4 y_{-2} + \cdots\cdots$$

$$f(1969) = 27 + 0.5(7) + \frac{(0.5)(1.5)}{2!}(5) + \frac{(0.5)(1.5)(-0.5)}{3!}(3) + \frac{(0.5)(0.5)(1.5)(2.5)}{4!}(7) + \frac{0.5(1.5)(-0.5)(2.5)(-1.5)}{5!}(-10)$$

$$= 27 + 3.5 + 1.875 - 0.1875 + 0.2743 - 0.1172$$

$$= 32.345$$

(Answer).

Question:

Find polynomial satisfying value

x: 2 : 4 : 6 : 8 : 10

y: -2 : 1 : 3 : 8 : 20

by using Gauss Formulae.

Solution:-

The difference table is

	x	y	Δ	Δ^3	Δ^4	Δ^5	
	2	-2					
			3				
	4	1		-1			
			$\boxed{2}$		$\boxed{4}$		
x_0	6	$\boxed{3}$		$\boxed{3}$		$\boxed{0}$	C.L
			$\boxed{5}$		$\boxed{4}$		
	8	8		7			
			12				
	10	20					

$$u = \frac{x - x_0}{h} = \frac{x-6}{2} \longrightarrow (i)$$

(i) Gauss Forward Formula:

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{1}{3!}u(u-1)(u+1)\Delta^3 y_{-1} + \frac{u(u-1)(u+1)(u-2)}{4!}\Delta^4 y_{-2} + \cdots\cdots$$

$$f(x) = 3 + u(5) + \frac{u(u-1)}{2}(3) + \frac{1}{6}u(u^2-1)(4) + 0$$

$$= \frac{2}{3}u^3 + \frac{3}{2}u^2 + \frac{17}{6}u + 3$$

using (i)

$$f(x) = \frac{2}{3}\frac{(x-6)^3}{8} + \frac{3}{2}\frac{(x-6)^2}{4} + \frac{17}{6}\frac{(x-6)}{2} + 3$$

$$= \frac{1}{12}(x^3 - 18x^2 + 108x - 216) + \frac{3}{8}(x^2 + 36 - 12x) + \frac{17}{12}(x-6) + 3$$

$$f(x) = \frac{1}{12}x^3 - \frac{9}{5}x^2 + \frac{71}{12}x - 10.$$

required (Ans.)

(ii) <u>Gauss Backward Formula:-</u>

$$f(x) = y_0 + u\Delta y_{-1} + \frac{1}{2}u(u+1)\Delta^2 y_{-1} + \frac{1}{3!}u(u+1)(u-1)\Delta^3 y_{-2} + \dots$$

Put $u = \frac{x-6}{2}$

$$f(x) = 3 + u(2) + \frac{1}{2}(u^2+u)(3) + \frac{1}{6}u(u^2-1)(4)$$

$$= 3 + 2u + \frac{3}{2}u^2 + \frac{3}{2}u + \frac{2}{3}u^3 - \frac{2}{3}u$$

$$f(x) = \frac{2}{3}u^3 + \frac{3}{2}u^2 + \frac{17}{6}u + 3$$

$$= \frac{2}{3}\frac{(x-6)^3}{8} + \frac{3}{2}\frac{(x-6)^2}{4} + \frac{17}{6}\frac{(x-6)}{2} + 3$$

$$f(x) = \frac{1}{12}x^3 - \frac{9}{8}x^2 + \frac{71}{12}x - 10$$

which is required polynomial.

(Answer).

EXERCISE

(1) Given Data, Find interpolation polynomial using
(i) Newton's forward and
(ii) Newton's backward Difference formula.
Also find $f(0.25)$

x	0.1	0.2	0.3	0.4	0.5
y	1.4	1.56	1.76	2.0	2.28

Ans: $2x^2 + x + 1.2$
$(f(0.25) = 1.655)$

(2) Estimate number of Students for 1953 from data given below

Year	1948	1950	1952	1954
No. of Students	50	79	102	113

(3) Find polynomial fitting following data

x	-3	-2	-1	0	1
y	16	7	4	1	-8

Hence calculate y corresponding $x = -1.5$

(4) Write $y = \sin\frac{\pi x}{2}$ as polynomial where Value consider with at $x = 0, 1, 2, 3$.

(5) Use Newton's Forward D. Formula to produce a third degree polynomial satisfy following data

x	0	2	4	6
$f(x)$	40	48	88	-224

(6) Interpolate by means of Gauss's Forward formula for $f(32)$ given

$f(25) = 0.2707$

$f(30) = 0.3027$

$f(35) = 0.3386$

$f(40) = 0.3794$

Ans: 0.3166

(7) The pressure of wind 'p' corresponding to Velocity 'v' is given by following data Estimate when $V = 25$ find p ?

V	10	20	30	40
p	1.1	2	4.4	7.9

Ans: 3.0375

(III) Stirling's Formula :-

$$f(x) = y_0 + u\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

where $u = \frac{x - x_0}{h}$

Proof:-

By Gauss's forward Formula

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1} + \cdots \longrightarrow (1)$$

By Gauss's backward Formula

$$f(x) = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{u(u+1)(u-1)}{3!}\Delta^3 y_{-2} + \frac{u(u+1)(u-1)(u+2)}{4!}\Delta^4 y_{-2} + \cdots \longrightarrow (2)$$

Taking Mean of (1) and (2)

$$f(x) = y_0 + u\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

Which is required formula.

* Question:-
If $y_{20} = 512$, $y_{30} = 439$, $y_{40} = 346$, $y_{50} = 243$ Find y_{35} by stirling Formula.

Solution:-

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_{-1}	20	512			
			-73		
x_0	30	439		-20	
			-93		10
x_1	40	346		-10	
			-103		
x_2	50	243			

$u = \frac{x - x_0}{h} = \frac{35 - 30}{10} = 0.5$

By stirling Formula:

$f(x) = y_0 + u\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1} + \dots\dots$

$= 439 + (0.5)\left(\frac{-73 - 93}{2}\right) + \frac{(0.5)^2}{2!}(-20)$

$= 439 - 41.5 - 2.3$

$f(x) = 395$

(Answer.)

* Question:- Using Stirling formula to find $y(28)$ given that

X	20	25	30	35	40
Y	49225	48316	47236	45926	44306

Solution:-
The difference table is

X	Y	Δ	Δ^2	Δ^3	Δ^4
20	49225				
		-909			
25	48316		-171		
		-1080		-65	
30	47236		-230		-9
		-1310		-74	
35	45926		-310		
		-1620			
40	44306				

By stirling formula $\quad u = \frac{x - 30}{5} = \frac{28 - 30}{5} = -0.4$

$f(x) = y_0 + u\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$

$+ \frac{1}{4!}u^2(u^2-1)\Delta^4 y_{-2}$

$= 47236 + (-0.4)\left(\frac{-1080 - 1310}{2}\right) +$

$\frac{(-0.4)^2}{2!}(-230) + \frac{(-0.4)((0.4)^2 - 1)}{3!} \cdot \left(\frac{-65 - 74}{2}\right)$

$+ \frac{1}{4!}(-0.4)^2(+0.160 - 1)(-9)$

$= 47236 + 478 - 18.4 - 3892 + 0.0504$

$f(28) \approx 47692$

(Answer).

Exercise

(i) Use stirling formula estimate $y(2.38)$ From table.

x	2.0	2.2	2.4	2.6	2.8
$f(x)$	-1.7843	-0.5537	0.0123	1.3976	1.9798

(ii) Using stirling formula find polynomial satisfy following data

x	20	30	40	50
$f(x)$	1313	1727	2392	3493

(iii) Using stirling formula find $\tan 16°$.

θ	0	$5°$	$10°$	$15°$	$20°$	$25°$	$30°$
$\tan\theta$	0	0.0875	0.1763	0.2679	0.3640	0.4663	0.5774

Ans: 0.2867

(IV) BESSEL'S FORMULA :-

$$f(x) = \frac{1}{2}(y_0 + y_1) + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left(\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2}\right)$$

$$+ \frac{u(u-1)(u-1/2)}{3!}\Delta^3 y_{-1} + \frac{u(u+1)(u-1)(u-2)}{4!} \cdot \left(\frac{\Delta^4 y_{-1} + \Delta^4 y_{-2}}{2}\right) + \cdots\cdots$$

PROOF:- As we know that Guass Forward Formula

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{u(u-1)(u+1)(u-2)}{4!}\Delta^4 y_{-2} + \cdots \longrightarrow (i)$$

And Guass Backward Formula is

$$f(x) = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{u(u+1)(u-1)}{3!}\Delta^3 y_{-2}$$

$$+ \frac{u(u+1)(u-1)(u+2)}{4!}\Delta^4 y_{-2} + \cdots\cdots \longrightarrow (ii)$$

Shifting origin from x_0 to x_1 in (ii). also u will be changed.

As $u = \frac{x - x_0}{h}$

$u = \frac{x - x_1}{h}$

$u = \frac{x - (x_0 + h)}{h} = \frac{x - x_0 - h}{h}$

$u = \frac{x - x_0}{h} - 1$

$u = u - 1$

Value of u will be changed to u - 1.
Put in (ii) we get

$$f(x) = y_1 + (u-1)\Delta y_0 + \frac{(u-1)u}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{u(u-1)(u-2)(u+1)}{4!}\Delta^4 y_{-1} + \ldots\ldots \longrightarrow \text{(iii)}$$

Taking mean of (i) and (iii)

$$f(x) = \frac{y_0 + y_1}{2} + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left(\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2}\right)$$

$$+ \frac{u(u-1)(u-\frac{1}{2})}{3!}\Delta^3 y_{-1} + \frac{u(u-1)(u+1)(u-2)}{4!} \cdot \left[\frac{\Delta^4 y_{-1} + \Delta^4 y_{-2}}{2}\right] + \ldots\ldots$$

which is required Bessel's formula.

<u>Question:-</u>
Let $y_{20} = 2854$, $y_{24} = 3162$, $y_{28} = 3544$, $y_{32} = 3992$. Solve by Bessel's formula taking origin at 24. Find y_{25}.

<u>Solution:</u>

	X	Y	ΔY	$\Delta^2 Y$	$\Delta^3 Y$	
x_{-1}	20	2854				
			308			
x_0	24	3162		74		C.L
			382		8	
x_1	28	3544		66		
			448			
x_2	32	3992				

$$u = \frac{x - x_0}{h}$$

$$u = \frac{25 - 24}{4} = \frac{1}{4} = 0.25$$

By Bessel's formula

$$f(x) = \frac{y_0 + y_1}{2} + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left(\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2}\right)$$

$$+ \frac{u(u-1)(u-\frac{1}{2})}{3!}\Delta^3 y_{-1} + \ldots\ldots\ldots$$

$$f(25) = \frac{3162 + 3544}{2} + \left(\frac{1}{4} - \frac{1}{2}\right)(382) + \frac{(0.25)(0.25-1)}{2!} \cdot$$

$$\left(\frac{74 + 66}{2}\right) + \frac{(0.25)(0.25-1)(\frac{1}{4} - \frac{1}{2})}{3!}(8)$$

$$f(25) = 3353 - 95.5 - 6.5625 - 0.0625$$
$$= 3250.875$$

$$\Rightarrow y_{25} = 3250.875$$

(Answer).

Question :-

Use Bessel's formula to find a polynomial satisfying values

x :	2	4	6	8	10	12
y :	-2	1	3	8	20	32

Solution :-

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
2	-2					
		3				
4	1		-1			
		2		4		
x_0 6	3		3		0	
		5		4		-11
x_1 8	8		7		-11	
		12		-7		
10	20		0			
		12				
12	32					

$$u = \frac{x - x_0}{h} = \frac{x-6}{2}$$

By Bessel's Formula

$$y = \frac{1}{2}(y_0 + y_1) + (u - \tfrac{1}{2})\Delta y_0 + \frac{1}{2!}u(u-1)\left(\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2}\right)$$
$$+ \frac{1}{3!}u(u-1)(u-\tfrac{1}{2})\Delta^3 y_{-1} + \frac{1}{4!}u(u-1)(u+1)(u-2)\left(\frac{\Delta^4 y_{-1} + \Delta^4 y_{-2}}{2}\right)$$
$$+ \frac{1}{5!}u(u-1)(u+1)(u-2)(u-\tfrac{1}{2})(\Delta^5 y_{-2}) + \ldots\ldots$$

using values

$$y = \frac{1}{2}(3+8) + (u - \tfrac{1}{2})(5) + \frac{1}{2!}u(u-1)\left(\frac{3+7}{2}\right) +$$
$$\frac{1}{3!}(u)(u-1)(u-\tfrac{1}{2})(4) + \frac{1}{4!}u(u-1)(u+1)(u-2)\frac{(0-11)}{2} +$$
$$\frac{1}{5!}u(u-1)(u+1)(u-2)(u-\tfrac{1}{2})(-11)$$

$$= \frac{11}{2} + 5u - \frac{5}{2} + \frac{5}{2}u^2 - \frac{5}{2}u + \frac{2}{3}\left(u^3 - \frac{3}{2}u^2 + \frac{1}{2}u\right)$$
$$- \frac{11}{48}(u^4 - 2u^3 - u^2 + 2u) - \frac{11}{120}\left(u^5 - \frac{5}{2}u^4 + \frac{5}{2}u^2 - u\right)$$

$$= \frac{11}{2} + 5u - \frac{5}{2} + \frac{5}{2}u^2 - \frac{5}{2}u + \frac{2}{3}u^3 - u^2 + \frac{1}{3}u -$$
$$\cancel{\frac{11}{48}u^4} + \frac{11}{24}u^3 + \cancel{\frac{11}{48}u^2} - \frac{11}{24}u - \frac{11}{120}u^5 + \cancel{\frac{11}{48}u^4}$$
$$- \cancel{\frac{11}{48}u^2} + \frac{11}{120}u$$

$$y = -\frac{11}{120}u^5 + \frac{9}{8}u^3 + \frac{3}{2}u^2 + \frac{37}{15}u + 3$$

using value of u.

$$y = \frac{-11}{120}\left(\frac{x-6}{2}\right)^5 + \frac{9}{8}\left(\frac{x-6}{2}\right)^3 + \frac{3}{2}\left(\frac{x-6}{2}\right)^2 +$$

$$\frac{37}{5}\left(\frac{x-6}{2}\right) + 3$$

$$= \frac{-11}{3840}\left[x^5 - 30x^4 + 36x^3 - 216x^2 + 6480x - 7776\right]$$

$$+ \frac{9}{64}\left[x^3 - 18x^2 + 108x - 216\right] + \frac{3}{8}(x^2 - 12x + 36)$$

$$+ \frac{37}{30}(x-6) + 3$$

$$= \frac{-11}{3840}x^5 + \frac{11}{128}x^4 - \frac{33}{320}x^3 + \frac{99}{16}x^2 - \frac{297}{16}x$$

$$+ \frac{891}{40} + \frac{9}{64}x^3 - \frac{81}{32}x^2 + \frac{243}{16}x - \frac{243}{8}$$

$$+ \frac{3}{8}x^2 - \frac{9}{2}x + \frac{27}{2} + \frac{37}{30}x - \frac{37}{5} + 3.$$

$$y = f(x) = \frac{-11}{3840}x^5 + \frac{11}{128}x^4 - \frac{57}{64}x^3 + \frac{129}{32}x^2$$

$$- \frac{797}{120}x + 1.$$

which is required polynomial.

(V) LAPLACE - EVERETT FORMULA :

$$f(x) = \left[\binom{u}{1}y_1 + \binom{u+1}{3}\Delta^2 y_0 + \binom{u+2}{5}\Delta^4 y_{-1} + \cdots\right]$$

$$\left[\binom{u-1}{1}y_0 + \binom{u}{3}\Delta^2 y_{-1} + \binom{u+1}{5}\Delta^4 y \ldots + \cdots\right]$$

where $u =$ [illegible]

($\binom{u}{1}$ denotes ^{u} [illegible]

Proof:-

Gauss's Forward [illegible] is

$$y_x = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\text{[illegible]} + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{u(u-1)(u+1)\text{[illegible]}}{4!}\Delta^4 y_{-2} + \cdots\cdots$$

It can be written as

$$y_x = y_0 + {}^uC_1\Delta y_0 + {}^uC_2\Delta^2 y_{-1} + {}^{u+1}C_3\Delta^3 y_{-1} +$$
$${}^{u+1}C_4\Delta^4 y_{-2} + {}^{u+2}C_5\Delta^5 y_{-2} + \cdots\cdots \longrightarrow (1)$$

Now

$$\Delta^{2k+1}y_{-k} = \Delta^{2k}(\Delta y_{-k})$$
$$= \Delta^{2k}(y_{-k+1} - y_{-k})$$

put $k = 0, 1, 2, \ldots\ldots$ in

$$\Delta^{2k+1}y_{-k} = \Delta^{2k}y_{-k+1} - \Delta^{2k}y_{-k}$$

$$\Delta y_0 = y_1 - y_0$$
$$\Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1}$$
$$\Delta^5 y_{-2} = \Delta^4 y_{-1} - \Delta^4 y_{-2}$$

and so on.

put value in ①

$$y_x = y_0 + {}^{u}C_1(y_1 - y_0) + {}^{u}C_2(\Delta^2 y_{-1}) + {}^{u+1}C_3(\Delta^2 y_0 - \Delta^2 y_{-1})$$
$$+ {}^{u+1}C_4 \Delta^4 y_{-2} + {}^{u+2}C_5(\Delta^4 y_{-1} - \Delta^4 y_{-2}) + \ldots\ldots$$

$$y_x = {}^{u-1}C_0\, y_0 + {}^{u}C_1\, y_1 - {}^{u}C_1\, y_0 - ({}^{u+1}C_3 - {}^{u}C_2)\Delta^2 y_{-1}$$
$$+ {}^{u+1}C_3 \Delta^2 y_0 - ({}^{u+2}C_5 - {}^{u+1}C_4)\Delta^4 y_{-2} +$$
$${}^{u+2}C_5 \Delta^4 y_{-1} + \ldots\ldots$$

$$y_x = -({}^{u}C_1 - {}^{u-1}C_0)y_0 + {}^{u}C_1\, y_1 - ({}^{u+1}C_3 - {}^{u}C_2)\Delta^2 y_{-1}$$
$$+ {}^{u+1}C_3 \Delta^2 y_0 + ({}^{u+2}C_5 - {}^{u+1}C_4)\Delta^2 y_{-2} +$$
$${}^{u+1}C_5 \Delta^4 y_{-1} + \ldots\ldots$$

As we know that

$${}^{n}C_r + {}^{n}C_{r+1} = {}^{n+1}C_{r+1}$$

$$\Rightarrow {}^{n+1}C_{r+1} - {}^{n}C_r = {}^{n}C_{r+1}$$

By using this identity

$$\therefore y_x = -{}^{u-1}C_1\, y_0 + {}^{u}C_1 y_1 - {}^{u}C_3 \Delta^2 y_{-1} + {}^{u+1}C_3 \Delta^2 y_0$$
$$- {}^{u+1}C_3 \Delta^2 y_{-1} + {}^{u+2}C_5 \Delta^4 y_{-1} - \ldots\ldots$$

$$y_x = \left[{}^{u}C_1 y_1 + {}^{u+1}C_3 \Delta^2 y_0 + {}^{u+2}C_5 \Delta^4 y_{-1} + \ldots\right] -$$
$$\left[{}^{u-1}C_1 y_0 + {}^{u}C_3 \Delta^2 y_{-1} + {}^{u+1}C_5 \Delta^4 y_{-2} + \ldots\right]$$

(Required Formula).

* Question :- Apply Everett-formula to obtain value of $y(25)$ Given $y(20) = 2854$, $y(24) = 3162$, $y(28) = 3544$, $y(32) = 3992$.

Solution :-

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	2854			
		308		
x_0 24	3162		74	
		382		−8
x_1 28	3544		66	
		448		
32	3992			

After

Before

By Laplace Everett Formula

$$y_x = \left[{}^{u}C_1 y_1 + {}^{u+1}C_3 \Delta^2 y_0 + {}^{u+2}C_5 \Delta^4 y_{-1} \ldots\right] - \left[{}^{u-1}C_1 y_0 + {}^{u}C_3 \Delta^2 y_{-1} + {}^{u+1}C_5 \Delta^4 y_{-2} + \ldots\right]$$

$$u = \frac{x - x_0}{h} = \frac{25-24}{4} = 0.25$$

$$= u\, y_1 + \frac{u(u+1)(u-1)}{3!}\Delta^2 y_0 - \frac{u-1}{1}\, y_0 - \frac{u(u-1)(u-2)}{3!}\Delta^2 y_{-1}$$

$$y_{25} = 25(3544) + \frac{(0.25+1)(0.25)(0.25-1)}{3!}(66) -$$

$(0.25-1)(3162) - \frac{(0.25)(0.25-1)(0.25-2)(79)}{6}$

$= 886 - 2.5781 + 2371.5 - 4.047$

$y_{.25} = 3250.9$

(Answer)

* **Question:-**

Apply Everett to following data to find polynomial which satisfies value of y corresponding to each argument.

x	0	1	2	3	4	5
y	0	-1	8	135	704	2375

Solution:- The difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0	0					
		-1				
1	-1		10			
		9		108		
x_0 2	8		118		216	
		127		324		120
x_1 3	135		442		336	
		569		660		
4	704		1102			
		1671				
5	2375					

Take $u = \frac{x - x_0}{h} = \frac{x-2}{1} = x - 2$

$y = [{}^{u}C_1 y_1 + {}^{u+1}C_3 \Delta^2 y_0 + {}^{u+2}C_5 \Delta^4 y_{-1} + \ldots\ldots] - [{}^{u-1}C_1 y_0 + {}^{u}C_3 \Delta^2 y_{-1} + {}^{u+1}C_5 \Delta^4 y_{-2} + \ldots]$

Using values

$y = u(135) + \frac{u(u+1)(u-1)}{3!}(442) + \frac{1}{5!}(u+2)(u+1)u \times (u-1)(u-2)(336) - (u-1)8 - \frac{u}{3!}(u-1)(u-2)(118)$

$- \frac{1}{5!}(u+1)(u)(u-1)(u-2)(u-3)(216)$

$= 135u + \frac{221}{3}(u^3 - u) + \frac{14}{5}(u^5 - 5u^3 + 4u) - 8u$

$+ 8 - \frac{59}{3}(u^3 - 3u^2 + 2u) - \frac{9}{5}(u^5 - 5u^4 + 5u^3 + 5u^2 - 6u)$

$= u^5 + 9u^4 + 31u^3 + 50u^2 + 36u + 8$

$f(x) = u^5 + 9u^4 + 31u^3 + 50u^2 + 36u + 8$

put $u = x - 2$

$f(x) = (x-2)^5 + 9(x-2)^4 + 31(x-2)^3 + 50(x-2)^2 + 36(x-2) + 8$

$f(x) = x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$
$+ 9(x^4 - 8x^3 + 24x^2 - 32 + 16) +$
$31(x^3 - 6x^2 + 12x - 8) + 50(x^2 + 4 - 4x)$
$+ 36x - 72 + 8$

$$f(x) = x^5 + (-10+9)x^4 + (40-72+31)x^3 + (-80+50+216-186)x^2 + (80-285+372+36-200)x + (-32+144-248-72+8+200)$$

$$f(x) = x^5 - x^4 - x^3$$

which is required polynomial

FOR

UNEQUALLY SPACED DATA

(I) AITKEN'S METHOD :-

First the table for Aitken's Method is given by

x_k	y_k				$x_i - x$
x_0	y_0				$x_0 - x$
x_1	y_1	$p_{0,1}(x)$			$x_1 - x$
x_2	y_2	$p_{0,2}(x)$	$p_{0,1,2}(x)$		$x_2 - x$
x_3	y_3	$p_{0,3}(x)$	$p_{0,1,3}(x)$	$p_{0,1,2,3}(x)$	$x_3 - x$

where

$$p_{0,k}(x) = \frac{1}{x_k - x_0} \begin{vmatrix} y_0 & x_0 - x \\ y_k & x_k - x \end{vmatrix}$$

$$p_{0,1,k}(x) = \frac{1}{x_k - x_1} \begin{vmatrix} p_{0,1}(x) & x_1 - x \\ p_{0,k}(x) & x_k - x \end{vmatrix}$$

$$p_{0,1,2,k}(x) = \frac{1}{x_k - x_2} \begin{vmatrix} p_{0,1,2}(x) & x_2 - x \\ p_{0,1,k}(x) & x_k - x \end{vmatrix}$$

(In above table $p_{0,1,2,3}(x)$ is required answer.)

Question:-

Apply Aitken's iteration to following data to find $f(2)$.

x	0	1	4	6
$f(x)$	1	-1	1	-1

Solution:

x	y	$p_{0,k}(x)$	$p_{0,1,k}(x)$	$p_{0,1,2,k}(x)$
0	1			
1	-1	$\frac{1(-1)-1(-2)}{-1-(-2)} = 3$		
4	1	$\frac{1(2)-1(-2)}{2-(-2)} = 1$	$\frac{-3(2)-1(-1)}{2-(-1)} = \frac{-5}{3}$	
6	-1	$\frac{1(4)-1(2)}{4-(-2)} = \frac{1}{3}$	$\frac{-3(4)-\frac{1}{3}(-1)}{4-(-1)} = \frac{-7}{3}$	$\frac{-\frac{5}{3}(4)-(\frac{-7}{3})(2)}{4-2} = -1$

where $x_0 - x = 0 - 2 = -2$

$x_1 - x = 1 - 2 = -1$

$x_2 - x = 4 - 2 = 2$

$x_3 - x = 6 - 2 = 4.$

$y(2) = f(2) = -1.$

(Answer).

Remark:-

$x_1 - x_0 = (x_1 - x) - (x_0 - x)$

(From previous table)

* Question:-

Apply Aitken's Method Find $f(3)$.

x	0	1	2	4
y	1	1	2	5

Solution:- $x = 3$

x	y	$p_{0,k}(x)$	$p_{0,1,k}(x)$	$p_{0,1,2,k}(x)$	$x_k - 3$
0	1				-3
1	1	$\frac{1}{1-0}\begin{vmatrix} 1 & -3 \\ 1 & -2 \end{vmatrix} = 1$			-2
2	2	$\frac{1}{2-0}\begin{vmatrix} 1 & -3 \\ 2 & -1 \end{vmatrix} = 5/2$	$\frac{1}{2-1}\begin{vmatrix} 1 & -2 \\ 5/2 & -1 \end{vmatrix} = 4$		-1
4	5	$\frac{1}{4-0}\begin{vmatrix} 1 & -3 \\ 5 & 1 \end{vmatrix} = 4$	$\frac{1}{4-1}\begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} = 3$	$\frac{1}{(4-2)}\begin{vmatrix} 4 & -1 \\ 3 & 1 \end{vmatrix} = \frac{7}{2}$	1

$\Rightarrow y(3) = \frac{7}{2}$ is required Answer.

* Question:-

Use Aitken's iteration to following data to find $f(3)$.

x	0	1	2	4	5
y	0	16	48	88	0

Solution:-

x	y	$P_{0,k}(x)$	$P_{0,1,k}(x)$	$P_{0,1,2,k}(x)$	$P_{0,1,2,3,k}(x)$	
0	0					-3
1	16	$\dfrac{\begin{vmatrix} 0 & -3 \\ 16 & -2 \end{vmatrix}}{-2+3} = 48$				-2
2	48	$\dfrac{\begin{vmatrix} 0 & -3 \\ 48 & -1 \end{vmatrix}}{-1+3} = 72$	$\dfrac{\begin{vmatrix} 48 & -2 \\ 72 & -1 \end{vmatrix}}{-1+2} = 96$			-1
4	88	$\dfrac{\begin{vmatrix} 0 & -3 \\ 88 & 1 \end{vmatrix}}{1+3} = 16$	$\dfrac{\begin{vmatrix} 48 & -2 \\ 66 & 1 \end{vmatrix}}{1+2} = 60$	$\dfrac{\begin{vmatrix} 96 & -1 \\ 60 & 1 \end{vmatrix}}{1+1} = 78$		1
5	0	$\dfrac{\begin{vmatrix} 0 & -3 \\ 0 & 2 \end{vmatrix}}{2+3} = 0$	$\dfrac{\begin{vmatrix} 48 & -2 \\ 0 & 2 \end{vmatrix}}{2+2} = 24$	$\dfrac{\begin{vmatrix} 96 & -1 \\ 24 & 2 \end{vmatrix}}{2+1} = 72$	$\dfrac{\begin{vmatrix} 78 & 1 \\ 72 & 2 \end{vmatrix}}{2-1} = 84$	2

(II) LAGRANGE'S INTERPOLATION FORMULA

Let $f(x_0), f(x_1), \ldots, f(x_n)$ be values of function $y = f(x)$ corresponding to arguments $x_0, x_1, x_2, \ldots, x_n$ not necessarily equally spaced, then...

$$f(x) = \frac{(x-x_1)(x-x_2)\ldots\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)} f(x_0) + \frac{(x-x_0)(x-x_2)\ldots\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots\ldots(x_1-x_n)} f(x_1) + \ldots\ldots\ldots + \frac{(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots\ldots(x_n-x_{n-1})} f(x_n).$$

Proof:- Suppose $f(x)$ is a polynomial of degree n.

Let

$$\begin{aligned} f(x) = {} & (x-x_1)(x-x_2)(x-x_3)\ldots\ldots(x-x_n)A_0 + \\ & (x-x_0)(x-x_2)(x-x_3)\ldots\ldots(x-x_n)A_1 + \\ & (x-x_0)(x-x_1)(x-x_3)(x-x_4)\ldots\ldots(x-x_n)A_2 + \\ & \ldots\ldots\ldots\ldots\ldots\ldots + \\ & (x-x_0)(x-x_1)(x-x_3)\ldots\ldots(x-x_{n-1})A_n \rightarrow (i) \end{aligned}$$

where A's are constant to be determined.

Put $x = x_0$

$$f(x_0) = (x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)A_0$$

$$\Rightarrow A_0 = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}$$

Similarly; Put $x = x_1$ we get

$$A_1 = \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}$$

Continuing in this way

$$A_n = \frac{f(x_n)}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}$$

Put all A's value in (i) we get

$$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} f(x_0) + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} f(x_1) + \frac{(x-x_0)(x-x_1)(x-x_3)\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)} f(x_2)$$

$$+ \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} f(x_n)$$

Which is Lagrange's Interpolation Formula.

* **Question:-** Find polynomial function $f(x)$ given that $f(0)=2$, $f(1)=3$, $f(2)=12$ and $f(3)=35$. Also find $f(5)$.

Solution:-

x	0	1	2	3
$f(x)$	2	3	12	35

By Lagrange's Formula:-

$$f(x) = \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)}(2) + \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)}(3)$$

$$+ \frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)}(12) + \frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)}(35)$$

$$= \frac{x^3-6x^2+11x-6}{-3} + \frac{3(x^3-5x^2+6x)}{2}$$

$$- 6(x^3-4x^2+3x) + \frac{35}{6}(x^3-3x^2+2x)$$

$f(x) = x^3+x^2-x+2$ Required polynomial

$$f(5) = 5^3+5^2-5+2$$
$$= 125+25-5+2$$
$$= 147$$

(Answer)

+ **Question:-**

The value of x & y are given as below

x : 5 6 9 11
y : 12 13 14 16

Find value of y when x = 10. using Lagrange's Formula for unequal interval.

Solution:- Given

x	5	6	9	11
y	12	13	14	16

The Lagrange's Formula is

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

At x = 10 using values we have

$$f(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(13) + \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}(14) + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}(16)$$

$$f(10) = \frac{-4}{-24}(12) - \frac{5}{+15}(13) - \frac{20}{-24}(14) + \frac{20}{60}(16)$$

$$f(10) = 2 - \frac{13}{3} + \frac{35}{3} + \frac{16}{3}$$

$$f(10) = 14.667$$

(Answer).

+ **Question:-**

Given $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$, $\log_{10} 661 = 2.8202$. Find $\log_{10} 656$.

Solution:- Let $f(x) = \log_{10}(x)$.

x	654	658	659	661
f(x)	2.8156	2.8182	2.8189	2.8202

Applying Lagrange's Formula

$$\log_{10}(656) = f(656) = \frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)}(2.8156) + \frac{(656-654)(656-659)(656-661)}{(658-654)(658-659)(658-661)}(2.8182) + \frac{(656-654)(656-658)(656-661)}{(659-654)(659-658)(659-661)}(2.8189) + \frac{(656-654)(656-658)(656-659)}{(661-654)(661-658)(661-659)}(2.8202)$$

$$f(656) = \frac{-30}{-100}(2.8156) + \frac{30}{12}(2.8182) + \frac{20}{-10}(2.8189) + \frac{12}{42}(2.8202)$$

$$= 0.6033 + 7.0455 - 5.6378 + 0.8058$$

$$f(656) = 2.8168$$

(Answer)

* **THEOREM:-**

Let $L_i(x)$ denotes the Lagrange's polynomial of degree less then or equal to n and

$$\prod_{i=0}^{n}(x) = (x-x_0)(x-x_1)\dots\dots(x-x_n)$$

$$\prod_{i=0}^{n}{}'(x_i) = (x_i-x_0)(x_i-x_1)\dots\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots\dots(x_i-x_n)$$

then

$$L_i(x) = \frac{\prod_{i=0}^{n}(x)}{(x-x_i)\prod_{i=0}^{n}{}'(x_i)} \qquad i=0,1,2,\dots,n.$$

Proof:-

From Lagrange's interpolation formula

$$L_i(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)}{(x_i-x_0)(x_i-x_1)\dots(x_i-x_{i-1})\dots(x_i-x_n)}$$

xing and $\div$ing by $(x-x_i)$ we have

$$L_i(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_{i-1})(x-x_i)(x-x_{i+1})\dots(x-x_n)}{(x-x_i)(x_i-x_0)(x_i-x_1)\dots(x_i-x_{i-1})\dots(x_i-x_n)}$$

$$= \frac{\prod_{i=0}^{n}(x)}{(x-x_i)\prod_{i=0}^{n}{}'(x)}$$

proved.

Theorem:-

With the meaning of symbol as in previous theorem, prove that

$$\sum_{i=0}^{n}\frac{\prod_{i=0}^{n}(x)}{(x-x_i)\prod_{i=0}^{n}{}'(x_i)} = 1 \quad \text{i-e} \quad \sum_{i=0}^{n} L_i(x) = 1$$

Proof:- We know that

$$L_i(x) = \frac{\prod_{i=0}^{n}(x)}{(x-x_i)\prod_{i=0}^{n}{}'(x_i)} \longrightarrow (1)$$

Now $\prod_{i=0}^{n}(x) = (x-x_0)(x-x_1)(x-x_2)\dots(x-x_n)$

$$\Rightarrow \frac{1}{\prod_{i=0}^{n}(x)} = \frac{1}{(x-x_0)(x-x_1)(x-x_2)\dots(x-x_n)}$$

$$= \frac{A_0}{x-x_0} + \frac{A_1}{x-x_1} + \dots + \frac{A_i}{x-x_i} + \dots + \frac{A_n}{x-x_n} \longrightarrow (2)$$

$$\Rightarrow \frac{1}{\prod_{i=0}^{n}(x)} = \sum_{i=0}^{n}\frac{A_i}{(x-x_i)} \longrightarrow (3)$$

xing (i) by $x - x_i$ on both sides

$$\frac{(x-x_i)}{(x-x_0)(x-x_1)\cdots(x-x_i)\cdots(x-x_n)} = \frac{A_0(x-x_i)}{(x-x_0)} + \frac{A_1(x-x_i)}{x-x_1} + \dots + \frac{A_i(x-x_i)}{x-x_i} + \dots\dots + \frac{A_n(x-x_i)}{x-x_n}$$

Putting $x = x_i$

$$\frac{1}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})(x_i-x_n)} = A_i$$

$$\Rightarrow A_i = \frac{1}{\prod_{i=0}^{n}{}'(x_i)}$$

Put in (3)

$$\frac{1}{\prod_{i=0}^{n}(x)} = \sum_{i=0}^{n} \frac{1}{(x-x_i)\prod_{i=0}^{n}{}'(x_i)}$$

$$\Rightarrow \sum_{i=0}^{n} \frac{\prod_{i=0}^{n}(x)}{(x-x_i)\prod_{i=0}^{n}{}'(x_i)} = 1.$$

From (i)

$$\sum_{i=0}^{n} L_i(x) = 1$$

(proved).

* Inverse Lagrange's Interpolation Formula :- If for a function f has values $f(x_0), f(x_1), \dots, f(x_n)$ at $x_0, x_1, \dots, x_n$ then for given $f(x)$

$$x = \frac{[f(x)-f(x_1)][f(x)-f(x_2)]\dots[f(x)-f(x_n)]}{[f(x_0)-f(x_1)][f(x_0)-f(x_2)]\dots[f(x_0)-f(x_n)]} x_0$$

$$+ \frac{[f(x)-f(x_0)][f(x)-f(x_2)]\dots[f(x)-f(x_n)]}{[f(x_1)-f(x_0)][f(x_1)-f(x_2)]\dots[f(x_1)-f(x_n)]} x_1$$

$$+ \dots\dots\dots +$$

$$\frac{[f(x)-f(x_0)][f(x)-f(x_1)]\dots[f(x)-f(x_{n-1})]}{[f(x_n)-f(x_0)][f(x_n)-f(x_1)]\dots[f(x_n)-f(x_{n-1})]} x_n$$

Example :-

Given

x	1	3	4	5
$f(x)$	6	8	15	20

and $f(x) = 18$ Find x.

Solution :-

Given that

x	1	3	4	5
$f(x)$	6	8	15	20

By using inverse Lagrange's formula:

$$x = \frac{(18-8)(18-15)(18-20)}{(6-8)(6-15)(6-20)}(1) + \frac{(18-6)(18-15)(18-20)}{(8-6)(8-15)(8-20)}(3) + \frac{(18-6)(18-8)(18-20)}{(15-6)(15-8)(15-20)}(4) + \frac{(18-6)(18-8)(18-15)}{(20-6)(20-8)(20-15)}(5)$$

$$x = \frac{(10)(3)(-2)}{(-2)(-9)(-14)}(1) + \frac{(12)(3)(-2)}{(2)(-7)(-12)}(3) + \frac{(12)(10)(-2)}{(9)(7)(-5)}(4) + \frac{(12)(10)(3)}{(14)(12)(5)}(5)$$

$$= 0.2381 - 1.2857 + 3.0476 + 2.1429$$

$$x = 4.1429$$

(Answer)

(III) NEWTON'S DIVIDED DIFFERENCE FORMULA

* **Divided Difference :-** Let $x_0, x_1, x_2, \ldots\ldots x_n$ be the arguments, which are not necessarily equally spaced and a function f has the values $f(x_0), f(x_1), \ldots, f(x_n)$ over these arguments respectively. Then the quantities

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\ldots\ldots \quad f[x_{n-1}, x_n] = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

are called I^{st} divided difference

And

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \text{ etc}$$

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} \text{ etc}$$

$$\vdots \qquad\qquad \vdots$$

$$f[x_0, x_1, \ldots, x_{n-1}, x_n] = \frac{f[x_1, x_2, \ldots x_n] - f[x_0, x_1, \ldots x_{n-1}]}{x_n - x_0} \text{ etc}$$

are respectively called 2^{nd}, 3^{rd}, ..., n^{th} divided differences.

§ **THEOREM:-** Prove the symmetric property of divided difference.

(OR)

Show that divided difference is independent of order of argument.

Proof:- For 1st order:

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

This can be written as

$$f[x_0, x_1] = \frac{f(x_1)}{x_1 - x_0} - \frac{f(x_0)}{x_1 - x_0}$$

$$= \frac{f(x_1)}{x_1 - x_0} + \frac{f(x_0)}{x_0 - x_1}$$

$$= \frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0}$$

$$\Rightarrow f[x_0, x_1] = \frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0} \longrightarrow (*)$$

Now

$$f[x_1, x_0] = \frac{f(x_1)}{x_1 - x_0} + \frac{f(x_0)}{x_0 - x_1} \quad (\because \text{by } * \text{ def})$$

$$= \frac{f(x_0)}{x_0 - x_1} - \frac{f(x_1)}{x_0 - x_1}$$

$$= f[x_0, x_1]$$

$$\Rightarrow f[x_0, x_1] = f[x_1, x_0]$$

For 2nd Order:

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$= \frac{1}{x_2 - x_0}\left[\frac{f(x_1)}{x_1 - x_2} + \frac{f(x_2)}{x_2 - x_1} - \frac{f(x_0)}{x_0 - x_1} - \frac{f(x_1)}{x_1 - x_0}\right] \quad (\text{by } *)$$

$$= \frac{1}{x_2 - x_0}\left[f(x_1)\left(\frac{1}{x_1 - x_2} - \frac{1}{x_1 - x_0}\right) + \frac{f(x_2)}{x_2 - x_1} - \frac{f(x_0)}{x_0 - x_1}\right]$$

$$= \frac{1}{x_2 - x_0}\left[f(x_1)\left(\frac{x_1 - x_0 + x_2 - x_1}{(x_1 - x_2)(x_1 - x_0)}\right) + \frac{f(x_2)}{x_2 - x_1} - \frac{f(x_0)}{x_0 - x_1}\right]$$

$$= \frac{1}{x_2 - x_0}\left[\frac{f(x_1)(x_2 - x_0)}{(x_1 - x_2)(x_1 - x_0)} + \frac{f(x_2)}{x_2 - x_1} - \frac{f(x_0)}{x_0 - x_1}\right]$$

$$= \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_0 - x_1)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} \longrightarrow (A)$$

Now by using above concept we write $f[x_1, x_2, x_0]$ as

$$f[x_1, x_2, x_0] = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} + \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)}$$

$$= \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} \longrightarrow (B)$$

By (A) and (B)

$$f[x_1, x_2, x_0] = f[x_0, x_1, x_2]$$

I^{st} Divided DIFFERENCE $= \dfrac{\Delta}{h} = \dfrac{\Delta}{1!\,h}$

2^{nd} divided difference $= \dfrac{\Delta^2}{2h^2} = \dfrac{\Delta^2}{2!\,h^2}$

3^{rd} divided difference $= \dfrac{\Delta^3}{6h^3} = \dfrac{\Delta^3}{3!\,h^3}$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

n^{th} divided difference $= \dfrac{\Delta^n}{n!\,h^n}$

Hence n^{th} divided difference of $f(x)$ is

$$f(x) = \frac{1}{n!\,h^n}\,[n^{th} \text{ forward difference of } f(x)]$$

$$= \frac{1}{n!\,h^n}\,[\text{constant}]$$

$$= \text{constant}.$$

And $(n+1)^{th}$ divided difference of $f(x)$ is

$$f(x) = \frac{1}{(n+1)!\,h^{(n+1)}}\,[(n+1)^{th} \text{ forward diff. of } f(x)]$$

$$= \frac{1}{(n+1)!\,h^{(n+1)}}\,(0)$$

$$= 0$$

Hence it is proved that n^{th} divided difference of polynomial of degree n is constant all all higher differences are zero.

(proved).

§ NEWTON's Divided DIFFERENCE FORMULA

Let a function f has values $f(x_0), f(x_1), \dots, f(x_n)$ at argument $x_0, x_1, \dots, x_n$ which are not necessarily equally spaced then

$$f(x) = f(x_0) + (x-x_0)f[x_0, x_1] + (x-x_0)(x-x_1)\,f[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2)f[x_0, x_1, x_2, x_3] + \dots + (x-x_0)(x-x_1)(x-x_2)\dots(x-x_{n-1})f[x_0, x_1, \dots, x_n]$$

Proof:- Consider $f[x, x_0] = \dfrac{f(x_0) - f(x)}{x_0 - x}$

$\Rightarrow (x_0 - x)f[x, x_0] = f(x_0) - f(x)$

$\Rightarrow f(x) = f(x_0) + (x - x_0)f[x, x_0] \rightarrow (i)$

Next

$$f[x, x_0, x_1] = \frac{f[x_0, x_1] - f[x, x_0]}{x_1 - x}$$

$\Rightarrow (x_1 - x)f[x, x_0, x_1] = f[x_0, x_1] - f[x, x_0]$

$\Rightarrow f[x, x_0] = f[x_0, x_1] + (x - x_1)f[x, x_0, x_1]$

by $f[x, x_0]$ value in (i)

$$f(x) = f(x_0) + (x-x_0)\{f[x_0, x_1] + (x-x_1)f[x, x_0, x_1]\}$$

$$= f(x_0) + (x - x_0)f[x_0, x_1] + (x-x_0)(x-x_1)\,f[x, x_0, x_1] \rightarrow (ii)$$

Also

$$f[x, x_0, x_1, x_2] = \frac{f[x_0, x_1, x_2] - f[x, x_0, x_1]}{x_2 - x}$$

$$\Rightarrow f[x, x_0, x_1] = f[x_0, x_1, x_2] + (x-x_2) f[x, x_0, x_1, x_2]$$

putting in (ii)

$$f(x) = f(x_0) + (x-x_0) f[x_0, x_1] + (x-x_0)(x-x_1) \cdot f[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2) \cdot f[x, x_0, x_1, x_2]$$

Continuing this way we obtain

$$f(x) = f(x_0) + (x-x_0) f[x_0, x_1] + (x-x_0)(x-x_1) f[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2) f[x_0, x_1, x_2, x_3] + \cdots\cdots + (x-x_0)(x-x_1)\cdots\cdots(x-x_{n-1}) f[x, x_0, x_1, \ldots, x_n]$$

Now as n^{th} divided difference of a polynomial of degree n is constant and all higher differences are zero, so being $(n+1)$st divided difference $f[x, x_0, x_1, \ldots, x_n] = 0$

Thus finially;

$$f(x) = f(x_0) + (x-x_0) f[x_0, x_1] + (x-x_0)(x-x_1) f[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2) f[x_0, x_1, x_2, x_3] + (x-x_0)(x-x_1)(x-x_2)(x-x_3) f[x_0, x_1, x_2, x_3, x_4] + \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1}) f[x_0, x_1, \ldots, x_{n-1}]$$

(proved).

Question:- The observed values of function are respectively 168, 120, 72 & 63 at four positions 3, 7, 9 and 10 of indp. Variable. What is best estimate you can give for value of function at at position 6 of indp. Variable.

Solution:- Divided Difference Table is

x	$f(x)$	I^{st} D.D	2^{nd} D.D	3^{rd} D.D
3	168			
		$\frac{120-168}{7-3} = -12$		
			$\frac{-24+12}{9-3} = -2$	
7	120			
		$\frac{72-120}{9-7} = -24$		$\frac{5+2}{10-3} = 1$
			$\frac{-9+24}{10-7} = 5$	
9	72			
		$\frac{63-72}{10-9} = -9$		
10	63			

Newton's d.d. Formula

$$f(x) = f(x_0) + (x-x_0) f[x_0, x_1] + (x-x_0)(x-x_1) f[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2) f[x_0, x_1, x_2, x_3] + \cdots\cdots$$

As $x = 6$, $f(x_0) = 168$

$$f(6) = 168 + (6-3)(-12) + (6-3)(6-7)(-2) + (6-3)(6-7)(6-9)(1)$$
$$= 168 - 36 + 6 + 9$$
$$= 147$$

(Answer)

* Question:-
Find polynomial satisfying data by Newton's divided difference formula

x :	0	1	2	3
$f(x)$:	1	2	11	34

Solution:- Divided Difference Table

x	$f(x)$	I^{st} D.D	2^{nd} D.D	3^{rd} D.D
0	1			
		$\frac{2-1}{1-0}=1$		
1	2		$\frac{9-1}{2-0}=4$	
		$\frac{11-2}{2-1}=9$		$\frac{7-4}{3-0}=1$
2	11		$\frac{23-9}{3-1}=7$	
		$\frac{34-11}{3-2}=23$		
3	34			

Newton's D.D Formula is

$$f(x)=f(x_0)+(x-x_0)f[x_0,x_1]+(x-x_0)(x-x_1)f[x_0,x_1,x_2]$$
$$+(x-x_0)(x-x_1)(x-x_2)f[x_0,x_1,x_2,x_3]+\dots\dots$$

$$f(x)=1+(x-0)(1)+(x-0)(x-1)(4)+$$
$$(x-0)(x-1)(x-2)(1)$$
$$=1+x+4x^2-4x+x(x^2-3x+2)$$
$$=1+x+4x^2-4x+x^3-3x^2+2x$$
$$=x^3+x^2-x+1$$

(Answer).

Note:- In this question, arguments are equally spaced. Hence we can find $f(x)$ using Newton's forward or backward formula.

* Question:- Apply Newton's D.D Formula to find no. of person getting 6 rupess from the following data.

Income per day (RS) :	3	5	7	8	10
No. of person :	180	154	120	110	90

Solution:- Divided difference Table is

x	y	I^{st} D.D	2^{nd} D.D	3^{rd} D.D	4^{th} D.D
3	180				
		$\frac{154-180}{5-3}=-13$			
5	154		$\frac{-17+13}{7-3}=-1$		
		$\frac{120-154}{7-5}=-17$		$\frac{2.33+1}{8-3}=0.67$	
7	120		$\frac{-10+17}{8-5}=2.33$		$\frac{-0.47-0.67}{10-3}=-0.16$
		$\frac{110-120}{8-7}=-10$		$\frac{0-2.33}{10-5}=-0.47$	
8	110		$\frac{-10+10}{10-7}=0$		
		$\frac{90-110}{10-8}=-10$			
10	90				

Using Newton's Divided Difference Formula given that $x=6$

$$f(x) = f(x_0) + (x-x_0) f[x_0, x_1] + (x-x_0)(x-x_1) f[x_0, x_1, x_2] + \cdots$$

$$f(6) = 180 + (6-3)(-13) + (6-3)(6-5)(-1) + (6-3)(6-5)(6-7)(0.67) + (6-3)(6-5)(6-7)(6-8)(-0.16)$$

$$= 135$$

No. of person getting Rs. 6 per day is 135

(Answer)

Exercise

(i) Find polynomial satisfying ordered pair $(-1, 33)$, $(0, 5)$, $(2, 9)$ and $(5, 1335)$ by N.D.D. formula.

Ans: $3x^4 - 5x^3 + 6x^2 - 14x + 5$

(ii) Find $\ln(9.2)$ From
$\ln(8.80) = 2.0794$
$\ln(9.0) = 2.1972$
$\ln(9.5) = 2.2513$ by newton D.D. formula.

(iii) The following tables is give. what is form of function

(i)

x	0	1	2	5
y	2	3	12	147

Ans: $x^3 - x + 2$

(ii)

x	0	1	4	5
$f(x)$	8	11	78	123

Ans: $\frac{1}{6}(x^3 + 24x^2 - 7x + 48)$

(iv) Find polynomial of lowest possible degree which assumes values 3, 12, 15, -2 when $x = 3, 2, 1, -1$ respectively by using newton's divided difference formula.

Ans: $x^3 - 9x^2 + 17x + 6$

NUMARICAL DIFFERENTIATION

It is the technique of evaluating $f'(x)$ or $f''(x)$ or $f'''(x)$ etc. when for a function $f(x)$, it is given that at some arguments $x_0, x_1, \ldots, x_n$ has values $f(x_0), f(x_1), \ldots, f(x_n)$.

CASE (1):-

In either of the following cases

1) To calculate $f'(x)$, $f''(x)$ or $f'''(x)$ etc when $x \in [x_0, x_n]$ or $x \notin [x_0, x_n]$.
2) To calculate $f'(x)$, $f''(x)$ or $f'''(x)$ etc when $x_0, x_1, \ldots, x_n$ are equally spaced or $x_0, x_1, \ldots, x_n$ are not equally spaced.
3) To calculate $f'(x)$, $f''(x)$, $f'''(x)$ etc when $x = x_k$ or $x \neq x_k$ for $0 \le k \le n$

Just calculat $f(x)$ by any appropriate Method (Like Newton's Formulae, Gauss Formulae, Lagrange's Method etc) and then by $f(x)$ find $f'(x)$, $f''(x)$, $f'''(x)$ etc.

Example:- Find $f'(x)$ and $f''(x)$ at $x = 1.5$

from table

x :	1	2	3	4
$f(x)$:	4	11	30	67

Solution:- As given

x	1	2	3	4
$f(x)$	4	11	30	67

By Lagrange Interpolation formula:

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_4)$$

$$f(x) = \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)}(4) + \frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)}(11)$$

$$+ \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)}(30) + \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)}(67)$$

After Simplification we have

$f(x) = x^3 + 3$

$f'(x) = 3x^2$ and $f''(x) = 6x$

$f'(1.5) = 3(1.5)^2$ $\quad f''(1.5) = 6(1.5)$

$= 6.75$ $\quad = 9.0$

(Answer)

- **2nd Method:**

Using Newton's Forward formula

x	$f(x)$	Δ	Δ^2	Δ^3
1	$\boxed{4}$			
		$\boxed{7}$		
2	11		$\boxed{12}$	
		19		$\boxed{6}$
3	30		18	
		37		
4	67			

$x_0 = 1$, $h = 1$

$u = \frac{x - x_0}{h} = x - 1$

By Newton's F. Formula

$$f(x) = f(x_0) + u\,\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0)$$

$$= 4 + (x-1)(7) + \frac{(x-1)(x-2)}{2!}(12) + \frac{(x-1)(x-2)(x-3)}{3!}(6)$$

$$= x^3 + 3$$

$f(x) = x^3 + 3$

$f'(x) = 3x^2$

$f'(1.5) = 3(1.5)^2 = 6.75$

and $f''(x) = 6x$

$f''(1.5) = 6(1.5)$

$= 9.0$

Ans

Note:

If argument are not equally spaced then construct Divided difference table and use divided difference formula.

Case (II):

Numarical Differentiation at a point included in the table of value of function and given arguments are equally spaced. In this case, two (formulae) or techniques are used which are

(i) Numarical Differentiation by Forward Difference operator

(ii) Numarical Differentiation by backward Difference operator.

(i) Numerical Differentiation by Forward Difference Operator:-

As by Shift operator

$$E f(a) = f(a+h)$$

$$= f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots$$

∵ by Taylor expension

$$= f(a) + hD f(a) + \frac{h^2}{2!} D^2 f(a) + \frac{h^3}{3!} D^3 f(a) + \ldots$$

$$= \left[1 + hD + \frac{h^2 D^2}{2!} + \frac{h^3 D^3}{3!} + \ldots\right] f(a)$$

$$E f(a) = e^{hD} f(a)$$

$$\Rightarrow E = e^{hD} \rightarrow (i)$$

But if we consider forward difference operator

$$E = 1 + \Delta \rightarrow (ii)$$

$\Rightarrow$ By (i) and (ii)

$$e^{hD} = 1 + \Delta$$

$$hD = \ln(1+\Delta) \quad \because \text{taking log on b.s}$$

$$hD = \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \ldots$$

$$D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \ldots\right]$$

$$D^n = \frac{1}{h^n}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \ldots\right]^n \rightarrow (*)$$

When put $n = 1$ in $(*)$ we get

$$D = \frac{1}{h}\left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \ldots\right)$$

$$f'(a) = \frac{1}{h}\left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \ldots\right) f(a) \rightarrow (1)$$

when $n = 2$

$$f''(a) = D^2 f(a)$$

$$= \frac{1}{h^2}\left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \ldots\right)^2 f(a)$$

$$= \frac{1}{h^2}\left(\Delta^2 - \frac{1}{2}\Delta^3 - \frac{1}{2}\Delta^3 + \frac{1}{4}\Delta^4 + \frac{1}{3}\Delta^4 + \frac{1}{3}\Delta^4 - \frac{1}{4}\Delta^5 - \frac{1}{6}\Delta^5 - \frac{1}{6}\Delta^5 - \frac{1}{4}\Delta^5 - \ldots\right) f(a)$$

$$= \frac{1}{h^2}\left(\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \ldots\right) f(a) \rightarrow (2)$$

when $n = 3$

$$f'''(a) = \frac{1}{h^3}\left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \ldots\right)^3 f(a)$$

$$= \frac{1}{h^3}\left(\Delta^3 - \frac{3}{2}\Delta^4 + \frac{7}{4}\Delta^5 - \frac{15}{18}\Delta^6 + \ldots\right) f(a) \rightarrow (3)$$

(1), (2) and (3) are 1^{st}, 2^{nd} and 3^{rd} derivatives of $f(x)$.

Example:-

Given data

x: 0 1 2 3 4 5

y: 3 4 19 84 259 628

Find $f'(1)$ and $f''(1)$.

Solution: The difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0	3					
		1				
→ 1	4		14			
		$\boxed{15}$		36		
2	19		$\boxed{50}$		24	
		65		$\boxed{60}$		0
3	84		110		$\boxed{24}$	
		175		84		
4	259		194			
		369				
5	628					

Here $a = 1$

The first derivative is given by

$$Df(a) = \frac{1}{h}\left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \dots\right) f(a)$$

$$f'(1) = \frac{1}{h}\left(\Delta f(1) - \frac{\Delta^2 f(a)}{2} + \frac{\Delta^3 f(a)}{3} - \frac{\Delta^4 f(a)}{4} + \dots\right)$$

$$= \frac{1}{1}\left[15 - \frac{1}{2}(50) + \frac{1}{3}(60) - \frac{1}{4}(24)\right]$$

$$f'(1) = 15 - 25 + 20 - 6$$

$$f'(1) = 4$$

The second derivative is given by

$$D^2 f(a) = \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \dots\right] f(a)$$

$$f''(1) = \frac{1}{1}\left[50 - 60 + \frac{11}{12}(24)\right]$$

$$= 50 - 60 + 22 = 12$$

$$f''(1) = 22$$

(Answer)

Example:-

Calculate first three derivatives from data

x: 0 1 2 3 4 5

y: 2 2 8 26 62 128

at $x = 2$

Solution:-

x	y	Δ	Δ^2	Δ^3	Δ^4
0	2				
		0			
1	2		6		
		6		6	
→ 2	8		12		0
		$\boxed{18}$		6	
3	26		$\boxed{18}$		0
		36		$\boxed{6}$	
4	62		24		
		60			
5	128				

At $x = 2$ first derivative is

$$Df(2) = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \dots\right] f(2)$$

$$= \frac{1}{1}\left[18 - \frac{1}{2}(18) + \frac{1}{3}(6)\right]$$

$$= 18 - 9 + 2$$

$$f'(2) = 11$$

The 2^{nd} derivative at $x = 2$.

$$f''(2) = \frac{1}{h^2}\left[\Delta^2 - \Delta^3\right] f(2)$$

$$= \frac{1}{1}\left[18 - 6\right]$$

$$= 12.$$

3^{rd} derivative at $x = 2$

$$f'''(x) = \frac{1}{h^3}\left[\Delta^3\right] f(2)$$

$$= \frac{1}{1}(6)$$

$$f'''(x) = 6$$

Answer

(iii) Numerical Differentiation by Backward Difference Operator :-

Consider

$$Ef(a) = f(a+h)$$

$$= f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \dots$$

$$= f(a) + hDf(a) + \frac{h^2D^2}{2!}f(a) + \frac{h^3D^3}{3!}f(a) + \dots$$

$$= \left(1 + hD + \frac{(hD)^2}{2!} + \frac{(hD)^3}{3!} + \dots\right) f(a)$$

$$Ef(a) = e^{hD} f(a)$$

$$\Rightarrow E = e^{hD}$$

But $E = (1-\nabla)^{-1}$ ($\because$ by backward difference operator.

$$\Rightarrow (1-\nabla)^{-1} = e^{hD}$$

$$\Rightarrow hD = (-1)\ln(1-\nabla)$$

$$= \nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \dots$$

$$D = \frac{1}{h}\left(\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \dots\right)$$

So in general;

$$D^n = \frac{1}{h^n}\left(\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \dots\right)^n$$

For $n = 1$

$$Df(a) = \frac{1}{h}\left(\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \dots\right) f(a)$$

For $n=2$

$$D^2 f(a) = \frac{1}{h^2}\left(\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \dots\right)^2 f(a)$$

$$= \frac{1}{h^2}\left(\nabla^2 + \frac{\nabla^4}{4} + \frac{\nabla^3}{2} + \frac{\nabla^4}{3} + \frac{\nabla^4}{3} + \frac{1}{4}\nabla^5 + \frac{1}{6}\nabla^5 + \frac{1}{6}\nabla^5 + \frac{1}{4}\nabla^5 + \dots\right]$$

$$= \frac{1}{h^2}\left[\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \frac{5}{6}\nabla^5 + \dots\right) f(a)$$

For $n = 3$.

$$D^3 f(a) = \frac{1}{h^3}\left(\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \frac{\nabla^5}{5} + \dots\right)^3 f(a)$$

$$= \frac{1}{h^3}\left(\nabla^3 + \left(\frac{1}{2}+1\right)\nabla^4 + \left(\frac{1}{3}+\frac{1}{2}+\frac{11}{12}\right)\nabla^5 + \left(\frac{1}{4}+\frac{1}{3}+\frac{11}{24}+\frac{5}{6}\right)\nabla^6 + \dots\right) f(a)$$

$$= \frac{1}{h^3}\left[\nabla^3 + \frac{3}{2}\nabla^4 + \frac{7}{4}\nabla^5 + \frac{15}{8}\nabla^6 + \dots\right] f(a)$$

<u>Remark:-</u>

If we wish to calculate the derivative of a function at a point which is given in beginning of table, we use forward difference formula and if we wish to calculate derivative near end points of the data, then we use some backward difference formula for differentiation.

<u>Example:</u>

Find $f'(4)$, $f''(4)$ from table

x	0	1	2	3	4	5
$f(x)$	3	4	19	84	259	628

<u>Solution:-</u>

The difference table is:

x	$f(x)$	∇	∇^2	∇^3	∇^4	∇^5
0	3					
		1				
1	4		14			
		15		36		
2	19		50		$\boxed{24}$	
		65		$\boxed{60}$		0
3	84		$\boxed{110}$		24	
		$\boxed{175}$		84		
4	259		194			
		369				
5	628					

Now Here $a = 4$

The first derivative of the function is given by.

$$Df(4) = \frac{1}{h}\left[\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \dots\right] f(4)$$

$$= \frac{1}{1}\left[\nabla f(4) + \frac{\nabla^2 f(4)}{2} + \frac{\nabla^3 f(4)}{3} + \frac{\nabla^4 f(4)}{4}\right]$$

$$= 1\left[175 + \frac{1}{2}(110) + \frac{1}{3}(60) + \frac{1}{4}(24)\right]$$

$f'(4) = 175 + 55 + 20 + 6$
$= 256$

2nd derivative of function at $x = 4$

$$f''(4) = \frac{1}{h^2}\left[\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \frac{5}{6}\nabla^5 \ldots\right] f(4)$$

$$= 1\left[110 + 60 + \frac{11}{12}(24)\right]$$

$$= 110 + 60 + 22$$

$$f''(4) = 192$$

Question:-

Compute first three derivative of $f(x)$ at $x = 2$ from data

x :	0	1	2	3	4	5
y :	-6	-10	-8	6	38	94.

Solution:- The difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4
0	-6				
		-4			
1	-10		6		
		2		6	
→ 2	-8		12		0
		14		6	
3	6		8		0
		32		6	
4	38	56	24		
5	94				

Here: $a = 2$
First derivative is given by

$$Df(a) = \frac{1}{h}\left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \ldots\right) f(a)$$

$$= \frac{1}{1}\left(\Delta f(2) - \frac{\Delta^2 f(a)}{2} + \frac{\Delta^3 f(2)}{3} - \frac{\Delta^4 f(4)}{4} + \ldots\right)$$

$$= 14 - \frac{1}{2}(18) + \frac{1}{3}(6)$$

$$= 14 - 9 + 2 = 7$$

2nd derivative is

$$D^2 f(a) = \frac{1}{h^2}\left[\Delta^2 \text{[illegible]}\right] f(a)$$

$$= \text{[illegible]} - \Delta^3 f(2) + \ldots]$$

$$= \text{[illegible]} - 18 = -4$$

3rd derivative is

$$D^3 f(a) = \frac{1}{h}\left[\Delta^3 - \frac{3}{2}\Delta^4 + \ldots\right] f(a)$$

$$f'''(2) = \frac{1}{1}\left[\Delta^3 f(2)\right]$$

$$f'''(2) = 6$$

(Answer)

Exercise

1) Find first three derivative of $f(x)$ at $x = 2.2$

x : 0 2 3 4

$f(x)$: -2 16 40 82

Ans: $f' = 19.52$

$f'' = 13.2$

$f''' = 6$

2) Using backward difference operator formula find three first derivative

x : 0 1 2 3 4 5

$f(x)$: 2 2 8 26 62 122

Ans: $f' = 9$, $f'' = 6$

$f''' = 0$

3) Find minimum value of $f(x)$ from table

x : 0.60 0.65 0.70 0.75

$f(x)$: 0.6221 0.6155 0.6138 0.6170

Ans: $x = 0.692$

$y_{min} = 0.6137$

4) Calculate $f'(x)$ and $f''(x)$ at $x = 1.2$ the value of $f(x) = Tan^{-1}(e^x)$ tabulated below

x	1	1.1	1.2	1.3	1.4
$Tan^{-1}(e^x)$	1.21828	1.24946	1.27824	1.30473	1.32902

5): Find first two derivative at $x = 4$

x : 1 2 3 4 5 6 7

$f(x)$: 4 11 20 67 128 219 346

Ans: 48, 24

6). Find radius of Curvature at 0.6 on curve defined by following

x	0.0	0.2	0.4	0.6	0.8	1.0	1.2
y	0.70	1.175	1.811	2.666	3.801	5.292	7.232

Hint: Find $y'(0.6)$ and $y''(0.6)$ and put in

$$\rho = \frac{\{1 + (y')^2\}^{3/2}}{y''}$$

Numerical Integration

Sometime it is either difficult or impossible to evaluate the definite integral by using well known analytic method e.g

$$\int_0^{\pi} \frac{x + \tan^{-1}(x)}{\ln(1+\sin x) + e^{-x^2}}\, dx$$

In such cases we use Numarical integration which gives us an approximation to the value of defined Integration. There are several method avaiable for numarical integration, but the most commonly used methods be Classified in two groups:

(i) Newton - Cote's Rule

(ii) Gauss:- Quadrature Rule.

Numarical Integration

- Newton Cotes Rule
  - when $n=1$ Trapezoidal Rule
  - when $n=2$ Simpson's 1/3 Rule
  - when $n=3$ Simpson's 3/8 Rule
  - when $n=4$ Boole's Rule
  - when $n=6$ Weddle's Rule.
- Gauss Quadrature Rule
  - (i) Two-point Formula
  - (ii) Three point Formula.

Where n is number of intervals.

(1) Newton-Cotes Rule :

Let $y = f(x)$ takes the values $y_0, y_1, \ldots, y_n$ or $(f(x_0), f(x_1), \ldots, f(x_n))$ equally spaced arguments at $x_0, x_1, x_2, \ldots, x_n$ resp.

Let $x_1 - x_0 = x_2 = x_1 = h$

Then

$x_1 = x_0 + h$

$x_2 = x_1 + h = x_0 + 2h$

$x_3 = x_2 + h = x_0 + 3h$

....

$x_n = x_0 + nh$

where $u = \frac{x - x_0}{h}$

$\Rightarrow x = x_0 + uh$

$dx = h\,du$

When $x = x_0$, $x = x_n$

$u = 0$ $\quad u = \frac{x_n - x_0}{h} = \frac{nh}{h} = n$

$$\text{Now} \int_{x_0}^{x_n} f(x)\,dx = \int_0^n f(x_0 + uh)\,h\,du$$

$$= h \int_0^n f(x_0 + uh)\,du$$

$$= h \int_0^n (1+\Delta)^u f(x_0)\,du \quad \because E^u f(x_0) = (1+\Delta)^u f(x_0)$$

$$= h \int_0^n \Big[f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0) + \ldots \Big] du$$

$\because$ by Newton's forward difference

$$= h\int_0^n \Big[y_0 + u\Delta y_0 + \frac{(u^2-u)}{2}\Delta^2 y_0 + \frac{(u^3-3u^2+2u)}{6}\Delta^3 y_0 + \frac{(u^4-6u^3+11u^2-6u)}{24}\Delta^4 y_0 + \ldots \Big] du$$

$$= h\Big[y_0 u\Big|_0^n + \Delta y_0 \frac{u^2}{2}\Big|_0^n + \frac{\Delta^2 y_0}{2}\left(\frac{u^3}{3} - \frac{u^2}{2}\right)\Big|_0^n + \frac{\Delta^3 y_0}{6}\left(\frac{u^4}{4} - u^3 + u\right)\Big|_0^n + \frac{\Delta^4 y_0}{24}\left(\frac{u^5}{5} - \frac{6u^4}{4} + \frac{11u^3}{3} - \frac{6u^2}{2}\right)\Big|_0^n + \ldots \Big]$$

$$= h\Big[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left[\frac{n^3}{3} - \frac{n^2}{2}\right]\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 - n^2\right)\Delta^3 y_0 + \frac{1}{24}\left(\frac{n^5}{5} - \frac{3}{2}n^4 + \frac{11}{3}n^3 - 3n^2\right)\Delta^4 y_0 + \ldots \Big]$$

This is called Newton Cote's formula.

(i) Trapezoidal Rule :-

In order to evaluate $\int_a^b f(x)\,dx$; Let us devide $[a, b]$ into 'n' subintervals each of length 'h' where $h = \frac{b-a}{n}$ ..., then

$P = \{x_0, x_1, x_2, \ldots, x_n\}$ is partition of $[a, b]$ with $x_k - x_{k-1} = h \;\; \forall k$

Now for all

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \dots + \int_{x_{n-1}}^{x_n} f(x)dx \rightarrow (*)$$

1^{st} we Consider $\int_{x_0}^{x_1} f(x)\,dx$.

In this case $f(x)$ is a polynomial of degree 1. So this has first forward difference is constant and all higher order differences are zero.

Now put $n=1$ in Newton Cote's formula

$$\int_{x_0}^{x_1} f(x) = h\left[y_0 + \frac{1}{2}\Delta y_0 + 0 + 0 + \dots 0\right]$$

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] \quad \because \Delta y_0 = y_1 - y_0$$

$$= \frac{h}{2}\left[2y_0 + y_1 - y_0\right]$$

$$= \frac{h}{2}(y_0 + y_1)$$

Similarly: $\int_{x_1}^{x_2} f(x)dx = \frac{h}{2}(y_1 + y_2)$

$$\int_{x_2}^{x_3} f(x)dx = \frac{h}{2}(y_2 + y_3)$$

....

$$\int_{x_{n-1}}^{x_n} f(x)dx = \frac{h}{2}(y_{n-1} + y_n)$$

Put in $(*)$ we get

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}\left(y_0 + 2(y_1 + y_2 + y_3 + \dots + y_{n-1}) + y_n\right]$$

Which is trapezoidal Rule.

(ii) Simpson's (1/3) Rule:-

In order to evaluate $\int_a^b f(x)dx$. Let divide $[a,b]$ into "n" subintervals where n should be even

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \dots + \int_{x_{n-2}}^{x_n} f(x)dx \rightarrow (*)$$

$(\because n=2)$

1^{st} we find $\int_{x_0}^{x_2} f(x)\,dx$

In this case $f(x)$ is polynomial of degree 2. so that's first and 2^{nd} forward difference are constant and higher order difference are zero.

Put $n=2$ in Newton Cote's formula

$$\int_{x_0}^{x_2} f(x)dx = h\left[2y_0 + \frac{4}{2}\Delta y_0 + \frac{1}{2}\left(\frac{8}{3} - \frac{4}{2}\right)\Delta^2 y_0 + 0 + 0\right]$$

$$= h\left[2y_0 + 2\Delta y_0 + \frac{1}{3}\Delta^2 y_0\right]$$

$$= h\left[2y_0 + 2(y_1 - y_0) + \frac{1}{3}(y_2 - 2y_1 + y_0)\right]$$

$$= h\left[2y_0 + 2y_1 - 2y_0 + \frac{1}{3}y_2 - \frac{2}{3}y_1 + \frac{1}{3}y_0\right]$$

$= \frac{h}{3} (6y_1 - y_2 - 2y_1 - y_0)$

$= \frac{h}{3} (y_0 + 4y_1 + y_2)$

Similarly: $\int_{x_2}^{x_4} f(x)dx = \frac{h}{3} (y_2 + 4y_3 + y_4)$

$\int_{x_4}^{x_6} f(x)dx = \frac{h}{3} (y_4 + 4y_5 + y_6)$

. . . .

$\int_{x_{n-2}}^{x_n} f(x)dx = \frac{h}{3} (y_{n-2} + 4y_{n-1} + y_n)$

So by (*)

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{3} [y_0 + 4(y_1 + y_3 + y_5 + \dots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \dots + y_{n-2}) + y_n]$$

This formula is called Simpson 1/3 Rule

(iii) Simpson's (3/8) Rule:-

In order to evaluate $\int_a^b f(x)dx$. Let us denotes (n is multiple of 3) and divide [a, b] into intervals.

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_3} f(x)dx + \int_{x_3}^{x_6} f(x)dx + \dots + \int_{x_{n-3}}^{x_n} f(x)dx \rightarrow (*)$$

Consider $\int_{x_0} f(x)dx$ to find that integral

Let n = 3 in Newton Cote's formula we have

$$\int_{x_0}^{x_3} f(x)dx = 3h [y_0 + \frac{3}{2} \Delta y_0 + \frac{3}{4} \Delta^2 y_0 + \frac{1}{8} \Delta^3 y_0]$$

$= \frac{3h}{8} [8y_0 + 12 \Delta y_0 + 6 \Delta^2 y_0 + \Delta^3 y_0]$

$= \frac{3h}{8} [8y_0 + 12(y_1 - y_0) + 6(y_2 - 2y_1 + y_0) + (y_3 - 3y_2 + 3y_1 - y_0)]$

$= \frac{3h}{8} [8y_0 + \cancel{12y_1} - 12y_0 + 6y_2 - \cancel{12y_1} + 6y_0 + 3y_1 + y_3 - 3y_2 - y_0]$

$= \frac{3h}{8} [y_0 + 3y_1 + 3y_2 + y_3]$

$\int_{x_0}^{x_3} f(x)dx = \frac{3h}{8} [y_0 + 3y_1 + 3y_2 + y_3]$

Similarly;

$\int_{x_3}^{x_6} f(x)dx = \frac{3h}{8} [y_3 + 3y_4 + 3y_5 + y_6]$

$\int_{x_6}^{x_9} f(x)dx = \frac{3h}{8} [y_6 + 3y_7 + 3y_8 + y_9]$

. . . .

$\int_{x_{n-3}}^{x_n} f(x)dx = \frac{3h}{8} [y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$

Put in (*) we have

$$\int_{x_0}^{x_n} f(x)dx = \frac{3h}{8}\left[y_0 + 3(y_1+y_2+y_4+\dots+y_{n-1}) + 2(y_3+y_6+y_9+\dots+y_{n-3}) + y_n\right]$$

Which is Simpson's (3/8) Rule.

(iv) Boole's Rule :-

To evaluate $\int_a^b f(x)dx$ we partitioned interval $[a,b]$ where $n=4$ (or 5 points)

$f(x)$ is a polynomial of degree 4 and all differences of order higher then 4 are zero

Put $n=4$ in Newton Cote's formula.

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_4} f(x)dx + \int_{x_4}^{x_8} f(x)dx + \dots + \int_{x_{n-4}}^{x_n} f(x)dx \rightarrow (*)$$

First find $\int_{x_0}^{x_4} f(x)\,dx$. (then put $n=4$ in N.C.F)

$$\int_{x_0}^{x_4} f(x)dx = 4h\left[y_0 + 2\Delta y_0 + \frac{5}{3}\Delta^2 y_0 + \frac{2}{3}\Delta^3 y_0 + \frac{7}{90}\Delta^4 y_0\right]$$

$$= 4h\left[y_0 + 2(y_1-y_0) + \frac{5}{3}(y_2-2y_1+y_0) + \frac{2}{3}(y_3-3y_2+3y_1-y_0) + \frac{7}{90}(y_4-4y_3+6y_2-4y_1+y_0)\right]$$

$$= \frac{2}{45}h\left[90y_0 + 18(y_1-y_0) + 150(y_2-2y_1+y_0) + 60(y_3-3y_2+3y_1-y_0) + 7(y_4-4y_3+6y_2-4y_1+y_0)\right]$$

$$= \frac{2}{45}h\left[7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4\right]$$

Similarly;

$$\int_{x_4}^{x_8} f(x)dx = \frac{2}{45}h\left[7y_4 + 32y_5 + 12y_6 + 32y_7 + 7y_8\right]$$

$$\dots\dots\dots$$

$$\int_{x_{n-4}}^{x_n} f(x)dx = \frac{2}{45}h\left[7y_{n-4} + 32y_{n-3} + 12y_{n-2} + 32y_{n-1} + 7y_n\right]$$

Put all in (*) we get

$$\int_{x_0}^{x_n} f(x)dx = \frac{2}{45}h\left[7(y_0+y_n) + 32(y_1+y_3+y_5+\dots+y_{n-1}) + 12(y_2+y_6+y_{10}+\dots+y_{n-2}) + 14(y_4+y_8+y_{12}+\dots+y_{n-4})\right]$$

Which is required Boole's Rule.

V Weddle's Rule:-

In order to evaluate $\int_a^b f(x)dx$. Let divide $[a,b]$ into subinterval $P = \{x_0, x_1, x_2, \ldots, x_n\}$

For all

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_6} f(x)dx + \int_{x_6}^{x_{12}} f(x)dx + \cdots + \int_{x_{n-6}}^{x_n} f(x)dx \longrightarrow (*)$$

First find $\int_{x_0}^{x_6} f(x)dx$.

In this case $f(x)$ is polynomial of degree 6 and all forward difference of greater the 6 are zero.

Put $n = 6$ in Newton Cote's formula

$$\int_{x_0}^{x_6} f(x)dx = h\left[6y_0 + 18\Delta y_0 + 27\Delta^2 y_0 + 24\Delta^3 y_0 + \frac{123}{10}\Delta^4 y_0 + \frac{33}{10}\Delta^5 y_0 + \frac{3}{10}\Delta^6 y_0\right]$$

$$= \frac{3h}{10}\left[20y_0 + 60\Delta y_0 + 90\Delta^2 y_0 + 80\Delta^3 y_0 + 41\Delta^4 y_0 + 11\Delta^5 y_0 + \Delta^6 y_0\right]$$

$$= \frac{3h}{10}\Big[20y_0 + 6(y_1 - y_0) + 90(y_2 - 2y_1 + y_0) + 8(y_3 - 3y_2 + 3y_1 - y_0) + 41(y_4 - 4y_3 + 6y_2 - 4y_1 + y_0) + 11(y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0) + (y_6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0)\Big]$$

$$= \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right].$$

Similarly:

$$\int_{x_6}^{x_{12}} f(x)dx = \frac{3h}{10}\left[y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}\right]$$

$$\cdots\cdots\cdots$$

$$\int_{x_{n-6}}^{x_n} f(x)dx = \frac{3h}{10}\left[y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n\right].$$

Put in $(*)$ we have

$$\int_{x_0}^{x_n} f(x)dx = \frac{3h}{10}\Big[(y_0 + y_2 + y_4 + y_8 + \cdots + y_n) + 2(y_6 + y_{12} + y_{18} + \cdots + y_{n-6}) + 5(y_1 + y_5 + y_7 + y_{11} + \cdots + y_{n-1}) + 6(y_3 + y_9 + y_{15} + \cdots + y_{n-3})\Big]$$

Which is required weddle's rule.

<u>Question</u>:- Evaluate $\int_0^1 \frac{1}{1+x^2}$ using
(i) Trapezoida rule with 5 points
(ii) Simpson's Rule 5 points.
(iii) Boole's Rule.
Compare result with exact value.

<u>Solution</u>:-

$a = 0, \quad b = 1, \quad n = 4$

$h = \frac{b-a}{h} = \frac{1-0}{4} = 0.25$

$f(x) = \frac{1}{1+x^2}$

	x_0	x_1	x_2	x_3	x_4
x	0	0.25	0.5	0.75	1
$f(x)$	1	0.941	0.8	0.64	0.5

(i) Trapezoidal Rule:-

$$\int_{x_0}^{x_4} f(x)\,dx = \frac{h}{2}\left(y_0 + 2(y_1 + y_2 + y_3) + y_4\right)$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{0.25}{2}\left[1 + 2(0.941 + 0.8 + 0.64) + 0.5\right]$$

$$= \frac{0.25}{2}(6.282) = 0.78275$$

(ii) Simpson's 1/3 Rule:-

$$\int_{x_0}^{x_4} f(x)\,dx = \frac{h}{3}\left[y_0 + 4(y_1 + y_3) + 2y_2 + y_4\right]$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{0.25}{3}\left[1 + 4(0.941 + 0.64) + 2(0.8) + 0.5\right]$$

$$= 0.78533$$

(iii) Boole's Rule:-

$$\int_{x_0}^{x_4} f(x)\,dx = \frac{2h}{45}\left[7(y_0 + y_4) + 32(y_1 + y_3) + 12(y_2)\right]$$

$$= \frac{2(0.25)}{45}\left[7(1 + 0.5) + 32(0.941 + 0.64) + 12(0.8)\right]$$

$$= 0.785466$$

<u>Exact Value</u>: $\int_0^1 \frac{1}{1+x^2}\,dx = \tan^{-1}(x)\Big|_0^1$

$= \tan^{-1}(1) - \tan^{-1}(0)$

$= \frac{\pi}{4}$

$= 0.78539$

Error in Simpson's Rule is smallest
So Simpson's Rule is better rule
other then rules.

(Answer)

Example:-

Apply simpson's 1/3 rule to find approx. value of π from $\int_0^1 \frac{1}{1+x^2} dx$ taking 4 intervals.

Solution:- $I = \int_0^1 \frac{1}{1+x^2} dx$

$= \tan^{-1}(x) \Big|_0^1$

$= \frac{\pi}{4} - 0$

$\Rightarrow \quad I = \pi/4$

$\Rightarrow \quad \pi = 4I$

Now we find approximate value of I using Simpson's 1/3 Rule.

Take $a = 0$

$b = 1$

$n = 4$

$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$

So table is

x :	0	0.25	0.50	0.75	1.0
f(x) :	1	0.9412	0.80	0.640	0.50

Now by Simpson's 1/3 Rule

$\int_0^1 \frac{1}{1+x^2} dx = \frac{h}{3} [f_0 + f_4 + 4(f_1 + f_3) + 2f_2]$

$= \frac{0.25}{3} [1 + 5 + 4(0.9412 + 0.6400) + 2(0.80)]$

$I = 0.7854$

So approximate value of π is

$\pi = 4I$

$= 4(0.7854)$

$= 3.1416$

(Answer).

Example: Evaluate $\int_0^{\pi/2} \sin x \, dx$ using boole's Rule.

Solution:-

Here $a = 0$, $b = \pi/2$

$f(x) = \sin x$

Take $n = 4$

$h = \frac{b-a}{n} = \frac{\pi/2 - 0}{4} = \frac{\pi}{8}$

So table is

x	f(x) = y
0	0
$\pi/8$	0.3827
$\pi/4$	0.7071
$3\pi/8$	0.9239
$\pi/2$	1

By Boole's Rule :-

$$\int_0^{\pi/2} \sin x\,dx = \frac{2h}{45}\left[7y_0+32y_1+12y_2+32y_3+7y_4\right]$$

$$= \frac{2}{45}(\pi/8)\left[7(0)+32(0.3827)+12(0.7071)+32(0.9239)+7(1)\right]$$

$$= \frac{\pi}{180}(57.2964)$$

$$\int_0^{\pi/2} \sin x\,dx = 1.000$$

(Answer)

Example :

Use Boole's formula and 3/8 Simpson Rule in Combination to evaluate integral $\int_0^{0.7} f(x)dx$ when $f(x)$ given by

x :	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
y :	0	0.001	0.399	0.893	0.1579	0.2449	0.3497	0.4705

Solution :

$a = 0$
$b = 0.7$
$h = 0.1$ (as from table)

The given function is

	x	y
x_0	0	0
x_1	0.1	0.001
x_2	0.2	0.399
x_3	0.3	0.893
x_4	0.4	0.1579
x_5	0.5	0.2449
x_6	0.6	0.3497
x_7	0.7	0.4705

In Simpson's $\frac{3}{8}$ Rule $n=3$ and in boole's Rule $n=4$.
So there are two possibilities
(i) From x_0 to x_3 apply Simpson's 3/8 and x_3 to x_7 apply Boole's Rule then add it.
(ii) From x_0 to x_4 apply Boole's Rule and x_4 to x_7 apply Simpson's 3/8 Rule then add it.

$$\int_0^{0.7} f(x)dx = \frac{2h}{45}\left[7y_0+32y_1+12y_2+32y_3+7y_4\right] + \frac{3h}{8}\left[y_4+3(y_5+y_6)+y_7\right]$$

$$= \frac{2(0.1)}{45}\left[7(0)+32(0.001)+12(0.399)+32(0.893)+7(0.1579)\right] + \frac{3(0.1)}{8}\left[0.1579+3(0.2449+0.3497)+0.4705\right]$$

$$\int_0^{0.7} f(x)\,dx = \frac{0.2}{45}(4.4737) + \frac{0.3}{8}(2.4122)$$
$$= 0.0199 + 0.0905$$
$$= 0.1104$$

(Answer)

Exercise

1): Evaluate $\int_0^1 (1+x+e^x)\,dx$ using trapizoidal rule for 8 - trapeziums. upto 5 decimal places.

2): Evaluate $\int_0^{\pi} \frac{\sin x}{x}\,dx$ for $n=6$ by Simpson's rule.

3): Integrate function $f(x)$ given by following table (i) using 5 pts Simpson's Rule (ii) Boole's Rule

x :	1	1.5	2.0	2.5	3.0
$f(x)$:	1	0.44	0.25	0.16	0.11

4): Evaluate $\int_0^1 \frac{1}{1+x^4}\,dx$ using (i) Trapezoidal Rule (ii) Simpson's 3/8 Rule

Compare result with exact value and which of two method is superior and why.

5): Use 7 points weddle rule to evaluate $\int_0^{\pi/2} \sin x\,dx$ using 6 digits accuracy.

(II) Gauss Quadrature Formula

Gauss investigated that in the quadrature formulae, the equally spaced arguments make the accuracy of quadrature formula Limited.

He derived a quadrature formula in which the Limit of integration are from -1 to $+1$, if not then these can be made by putting

$$x = \frac{(b-a)t + (a+b)}{2}$$

$$\Rightarrow t = \frac{2x-a-b}{b-a}$$

at $x = a$, $t = \frac{2a-a-b}{b-a} = \frac{a-b}{b-a} = -1$

at $x = b$, $t = \frac{2b-a-b}{b-a} = \frac{b-a}{b-a} = +1$

§ Derivation of Gauss Two point Quadrature Formula :-

Let $I = \int_a^b f(x)\,dx$

Put $x = \frac{(b-a)t + (b+a)}{2}$

$$dx = \frac{b-a}{2}\,dt$$

when $x = a$, $t = -1$

$x = b$, $t = 1$

In this case $I = \int_a^b f(x)\,dx = \int_{-1}^{1} f\left(\frac{(b-a)t + (b+a)}{2}\right)\left(\frac{b-a}{2}\right)dt$

$$= \frac{b-a}{2}\int_{-1}^{1} F(t)\,dt \longrightarrow (i)$$

where

$$F(t) = f\left(\frac{(b-a)t + (b+a)}{2}\right)$$

Let $\int_{-1}^{1} F(t)\,dt = C_1 F(t_1) + C_2 F(t_2) \longrightarrow (ii)$

Two points are t_1 and t_2.

where C_1, C_2, t_1 and t_2 are unknown and $C_1 \neq 0$, $C_2 \neq 0$ and $t_1 \neq t_2$.

Now as, cubic polynomial can be constructed at the most by using (4) unknown quantites so, to determine 4 unknown quantites.

We take $F(t) = 1, t, t^2, t^3$

when $F(t) = 1$: then

$$\int_{-1}^{1} F(t)\,dt = C_1 F(t_1) + C_2 F(t_2)$$

$$\Rightarrow \int_{-1}^{1} 1\,dt = C_1(1) + C_2(1)$$

$$\boxed{2 = C_1 + C_2} \longrightarrow (iii)$$

When $F(t) = t$: then (ii) gives

$$\int_{-1}^{1} t\,dt = C_1 t_1 + C_2 t_2$$

$$\Rightarrow \boxed{0 = C_1 t_1 + C_2 t_2} \longrightarrow (iv)$$

When $F(t) = t^2$; then (ii) gives

$$\int_{-1}^{1} t^2 dt = C_1 t_1^2 + C_2 t_2^2$$

$$\left.\frac{t^3}{3}\right|_{-1}^{1} = C_1 t_1^2 + C_2 t_2^2$$

$$\boxed{\frac{2}{3} = C_1 t_1^2 + C_2 t_2^2} \longrightarrow (v)$$

When $F(t) = t^3$: then (ii) gives

$$\int_{-1}^{1} t^3 dt = C_1 t_1^3 + C_2 t_2^3$$

$$\left.\frac{t^4}{4}\right|_{-1}^{1} = C_1 t_1^3 + C_2 t_2^3$$

$$\Rightarrow \boxed{0 = C_1 t_1^3 + C_2 t_2^3} \longrightarrow (vi)$$

Now multiplying (4) by t_1^2 and then subtracting from eq (6)

$$\cancel{C_1 t_1^3} + C_2 t_2^3 = 0$$
$$\pm \cancel{C_1 t_1^3} \pm C_2 t_2 t_1^2 = 0$$
$$C_2 t_2 (t_2^2 - t_1^2) = 0$$
$$C_2 t_2 (t_2 - t_1)(t_2 + t_1) = 0.$$

$\Rightarrow C_2 = 0$ or $t_2 = 0$ or $t_2 = t_1$ or $t_2 = -t_1$

but $C_2 \neq 0$ and $t_2 \neq t_1$

So $t_2 = 0$ or $t_2 = -t_1$

When $t_2 = 0$ from (iv) $C_1 t_1 = 0$

$\Rightarrow C_1 = 0$ or $t_1 = 0$

but $C_1 \neq 0$

So $t_1 = 0$

But $t_1 = t_2$ which is not possible.

Hence $t_2 \neq 0$.

Hence finally $t_2 = -t_1$

From (v) $C_1 t_1^2 + C_2 (-t_1)^2 = 2/3$

$$\Rightarrow (C_1 + C_2) t_1^2 = 2/3$$

$$\Rightarrow 2 t_1^2 = \frac{2}{3} \quad (\because \text{by (iii)})$$

$$t = \pm 1/\sqrt{3}$$

When $t_1 = \frac{-1}{\sqrt{3}}$ then $t_2 = \frac{1}{\sqrt{3}}$

When $t_2 = \frac{-1}{\sqrt{3}}$ then $t_1 = \frac{1}{\sqrt{3}}$

Next Put $t_2 = -t_1$ in (iv)

$$C_1 t_1 - C_2 t_1 = 0$$
$$(C_1 - C_2) t_1 = 0$$
$$\Rightarrow C_1 - C_2 = 0 \quad \because t_1 \neq 0$$
$$\Rightarrow C_1 = C_2$$

From (iii) $C_1 + C_2 = 2$

$$2C_1 = 2$$
$$C_1 = 1$$
$$\Rightarrow C_2 = 1.$$

So from (ii)

$$\int_{-1}^{1} F(t) dt = (1) F\left(\frac{1}{\sqrt{3}}\right) + F\left(-\frac{1}{\sqrt{3}}\right)$$

Hence $I = \int_a^b F(x) dx = \frac{b-a}{2}\left[F\left(\frac{1}{\sqrt{3}}\right) + F\left(\frac{-1}{\sqrt{3}}\right)\right]$

Which is required Gauss two point Quadrature formula.

(proved)

* Question:- Evaluate $\int_0^{\pi/2} \sin x \, dx$ by two-point Gaussion formula.

Solution:- $I = \int_0^{\pi/2} \sin x \, dx$

$a = 0, \quad b = \pi/2$

Put $x = \dfrac{(b-a)t + (b+a)}{2}$

$$x = \frac{(\pi/2 - 0)t + (\pi/2 + 0)}{2}$$

$$x = \frac{\pi/2 \, t + \pi/2}{2}$$

$$dx = \frac{\pi}{4} dt$$

when $x = 0$; $t = -1$

when $x = \frac{\pi}{2}$; $t = 1$

$$I = \int_{-1}^{1} \sin\left(\frac{\pi/2 \, t + \pi/2}{2}\right) \frac{\pi}{4} dt$$

$$= \frac{\pi}{4} \int_{-1}^{1} F(t) \, dt \quad \text{where } F(t) = \sin\left(\frac{\frac{\pi}{2}t + \frac{\pi}{2}}{2}\right)$$

$$I = \frac{\pi}{4} \left[F\left(\frac{-1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right) \right]$$

$$= \frac{\pi}{4} \left[\sin\left(\frac{\pi/2(-1/\sqrt{3}) + \pi/2}{2}\right) + \sin\left(\frac{\pi/2(1/\sqrt{3}) + \pi/2}{2}\right) \right]$$

$$= \frac{\pi}{4} \left[\sin(0.3319) + \sin(1.2388) \right]$$

$$= \frac{\pi}{4} \left[0.3258 + 0.94541 \right]$$

$$= 0.99848$$

* Question:- Evalu[illegible] dx using two-point Guass-quad formula.

Solution: $I = \int_0^1 e^{-x} dx$

where $a = 0$, $b = 1$

Put $x = \dfrac{(b-a)t + (b+a)}{2}$

$$x = \frac{1t + 1}{2}$$

$$x = \frac{t+1}{2}$$

$$dx = \frac{1}{2} dt$$

When $x = 0$; $t = -1$

When $x = 1$; $t = 1$

$$I = \int_{-1}^{1} e^{-\left(\frac{t+1}{2}\right)} \cdot \frac{1}{2} \cdot dt$$

$$= \frac{1}{2} \int_{-1}^{1} e^{-\left(\frac{t+1}{2}\right)} dt$$

By Gauss quadrature formula

$$I = \frac{1}{2}\int_{-1}^{1} F(t)\,dt \quad \text{where} \quad F(t) = e^{-\left(\frac{t+1}{2}\right)}$$

$$= \frac{1}{2}\left[F\left(\frac{-1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$= \frac{1}{2}\left[e^{-\left(\frac{-1/\sqrt{3}+1}{2}\right)} + e^{-\left(\frac{1/\sqrt{3}+1}{2}\right)}\right]$$

$$= \frac{1}{2}\left[e^{-0.2113} + e^{-0.7887}\right]$$

$$= \frac{1}{2}\left[0.8095 + 0.4544\right]$$

$I = 0.6320$ (Answer).

* **GAUSS 3-point Quardrature Formula:**

(without prove)

$$\int_{-1}^{1} F(x)\,dx = \frac{1}{9}\left[5F\left(-\frac{\sqrt{3}}{\sqrt{5}}\right) + 8F(0) + 5F\left(\frac{\sqrt{3}}{\sqrt{5}}\right)\right]$$

(Not include its proof.)

Question:-

Apply Gauss Quadrature for 3-pts evaluate $\int_{-1}^{1} \frac{1}{1+x^2}\,dx$.

Solution:-

$$I = \int_{-1}^{1} \frac{1}{1+x^2}\,dx$$

$$= \int_{-1}^{1} F(x)\,dx \; ; \quad \text{where} \quad F(x) = \frac{1}{1+x^2}$$

$$I = \frac{1}{9}\left[5F\left(-\frac{\sqrt{3}}{\sqrt{5}}\right) + 8F(0) + 5F\left(\frac{\sqrt{3}}{\sqrt{5}}\right)\right]$$

$$= \frac{1}{9}\left[5 \cdot \frac{1}{1+3/5} + 8(1) + 5 \cdot \frac{1}{1+3/5}\right]$$

$$= \frac{1}{9}\left[5(0.625) + 8(1) + 5(0.625)\right]$$

$$= \frac{1}{9}(14.25)$$

$I = 1.5833$ (Ans)

* Question:

Apply 3-pt Gaussian Formula for

$$I = \int_2^3 \frac{1}{2-\sin x^2}\, dx$$

Solution:- $I = \int_2^3 \frac{1}{2-\sin x^2}\, dx$

where $a = 2$, $b = 3$

put $x = \frac{(b-a)t+(a+b)}{2}$

$$x = \frac{(3-2)t+5}{2}$$

$$x = \frac{1t+5}{2}$$

$$dx = \frac{1}{2}\, dt$$

when $x = 2$; $t = -1$

when $x = 3$; $t = 1$

$$I = \int_{-1}^{1} \frac{1}{2-\sin\left(\frac{t+5}{2}\right)^2}\, \frac{1}{2}\, dt$$

$$= \frac{1}{2}\int_{-1}^{1} \frac{1}{2-\sin\left(\frac{t+5}{2}\right)^2}\, dt \longrightarrow (*)$$

$$\int_{-1}^{1} \frac{1}{2-\sin\left(\frac{t+5}{2}\right)^2}\, dt = \frac{1}{9}\left[F\left(-\sqrt{\frac{3}{5}}\right).5 + 8F(0) + 5F\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{1}{9}\left[\frac{1\cdot 5}{2-\sin\left(\frac{-\sqrt{3/5}+5}{2}\right)^2} + 8\left(\frac{1}{2}\right) + \frac{5}{2-\sin\left(\frac{\sqrt{3/5}+5}{2}\right)^2}\right]$$

$$= \frac{1}{9}\left[\frac{5}{2-0.7340} + \frac{8}{2} + \frac{5}{2-0.0633}\right]$$

$$= \frac{1}{9}\left(3.9494 + 4 + 2.5817\right)$$

$$= \frac{10.5311}{9}$$

$$= 1.1701$$

put in *

$$I = \frac{1}{2}\int_{-1}^{1} \frac{1}{2-\sin\left(\frac{t+5}{2}\right)^2}\, dt$$

$$= \frac{1}{2}(1.1701) = 0.5851$$

(Answer)

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Exercise:

Apply 2 and 3 pts Gaussian Formulae.

(i) $I = \int_0^1 \frac{dx}{1+x^2}$ Ans: 0.7868

(ii) $I = \int_0^1 \frac{\sin x}{x}\, dx$ Ans: 0.9443

(iii) $I = \int_0^1 x\, dx$ Ans: 0.5

(iv) $I = \int_{-1}^{1} (1-x^2)^{2/3}\cos x\, dx$ Ans: 1.32.

ERROR ANALYSIS

In Numerical Integration; we study many methods to evaluate problem and we demonstrated through examples that some methods are superior to others but we did not give any theoretical justification.

Now we shall calculate the amount of error in each method, for this we shall use Taylor's Series.

Note:

Error = X − A where

X = Exact value

A = Approximate value

(1) Error Analysis of Trapezoidal Rule ; $n=1$

$$E = X - A$$

$$= \int_{x_0}^{x_1} f(x)dx - \frac{h}{2}(f_0 + f_1)$$

$$= \int_{x_0}^{x_1} f(x)dx - \frac{h}{2}\left[f_0 + f(x_0+h)\right]$$

$$= F(x_1) - F(x_0) - \frac{h}{2}\left[f_0 + f(x_0+h)\right]$$

$$= F(x_0+h) - F(x_0) - \frac{h}{2}\left[f_0 + f(x_0+h)\right]$$

$$= \cancel{F(x_0)} + hF'(x_0) + \frac{h^2}{2!}F''(x_0) + \ldots - \cancel{F(x_0)}$$

$$- \frac{h}{2}\left[f_0 + \left(f_0 + hf_0' + \frac{h^2}{2!}f_0'' + \ldots\right)\right]$$

$$= \cancel{h}f_0 + \cancel{\frac{h^2}{2!}f_0'} + \frac{h^3}{3}f_0'' + \ldots - \cancel{\frac{h}{2}}f_0 - \cancel{\frac{h}{2}}f_0$$

$$- \cancel{\frac{h^2}{2}f_0'} - \frac{h^3}{2.2!}f_0'' - \frac{h^4}{2.3!}f_0''' - \ldots$$

$$= \left(\frac{1}{3!} - \frac{1}{2.2!}\right)h^3 f_0'' + \ldots$$

$$= \left(\frac{1}{6} - \frac{1}{4}\right)h^3 f_0''$$

$$E = -\frac{1}{12}h^3 f_0'' \quad \text{or} \quad -\frac{1}{12}h^3 f''(x_0)$$

(II) Error Analysis of Simpson's $\frac{1}{3}$ Rule ; $n = 2$.

$$E = X - A$$
$$= \int_{x_0}^{x_2} f(x)\,dx - \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right)$$
$$E = I_1 - I_2 \longrightarrow (i)$$
$$I_1 = \int_{x_0}^{x_2} f(x)\,dx \qquad \text{Let } f(x) = F'(x)$$
$$= \int_{x_0}^{x_2} F'(x)\,dx$$
$$= \Big| F(x) \Big|_{x_0}^{x_2}$$

$$= F(x_2) - F(x_0)$$
$$= F(x_0 + 2h) - F(x_0)$$
$$= \cancel{F(x_0)} + 2h F'(x_0) + \frac{4h^2}{2!} F''(x_0) + \frac{8h^3}{3!} F'''(x_0) + \dots - \cancel{F(x_0)}$$
$$I_1 = 2h f(x_0) + 2h^2 f'(x_0) + \frac{4}{3} h^3 f''(x_0) + \frac{2}{3} h^4 f'''(x_0) + \frac{4}{15} h^5 f^{iv}(x_0) + \dots$$
$$I_2 = \frac{h}{3}\left[f(x_0) + 4 f(x_1) + f(x_2) \right]$$
$$= \frac{h}{3}\left[f(x_0) + 4 f(x_0 + h) + f(x_0 + 2h) \right]$$
$$= \frac{h}{3}\Big[f(x_0) + 4\Big\{ f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + \dots \Big\} + f(x_0) + 2h f'(x_0) + \frac{4h^2}{2!} f''(x_0) + \frac{8h^3}{3!} f'''(x_0) + \dots \Big]$$

$$= 2h f(x_0) + 2h^2 f'(x_0) + \frac{4}{3} h^3 f''(x_0) + \frac{2}{3} h^4 f'''(x_0) + \frac{5}{18} h^5 f^{iv}(x_0) + \dots$$

Put I_1 and I_2's value in (i)

$$E = \frac{-1}{90} h^5 f^{iv}(x_0) \qquad \because \text{by neglecting higher terms.}$$

(III) Error Analysis of Simpson's $\frac{3}{8}$ Rule ; $n = 3$.

$$E = X - A$$
$$= \int_{x_0}^{x_3} f(x)\,dx - \frac{3h}{8}\left\{ f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3) \right\}$$
$$= I_1 - I_2 \longrightarrow (ii)$$
$$I_1 = \int_{x_0}^{x_3} f(x)\,dx \qquad \text{let } f(x) = F'(x)$$
$$= \int_{x_0}^{x_3} F'(x)\,dx$$
$$= F(x)\Big|_{x_0}^{x_3} = F(x_3) - F(x_0)$$
$$= F(x_0 + 3h) - F(x_0)$$
$$= \cancel{F(x_0)} + 3h F'(x_0) + \frac{(3h)^2}{2!} F''(x_0) + \frac{(3h)^3}{3!} F'''(x_0) + \frac{(3h)^4}{4!} F^{iv}(x_0) + \dots - \cancel{F(x_0)}$$

$$I = 3hf(x_0) + \frac{9h^2}{2}f'(x_0) + \frac{9h^3}{2}f''(x_0) + \frac{27}{8}h^4 f'''(x_0) + \frac{81}{40}h^5 f^{iv}(x_0) + \dots$$

And

$$I_2 = \frac{3h}{8}\left[f(x_0) + 3f(x_0+h) + 3f(x_0+2h) + f(x_0+3h)\right]$$

$$= \frac{3h}{8}\Big[f(x_0) + 3\Big\{f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \frac{h^3}{3!}f'''(x_0) + \frac{h^4}{4!}f^{iv}(x_0) + \dots\Big\}$$

$$+ 3\Big\{f(x_0) + 2hf'(x_0) + \frac{(2h)^2}{2!}f''(x_0) + \frac{(2h)^3}{3!}f'''(x_0) + \frac{(2h)^4}{4!}f^{iv}(x_0) + \dots\Big\} +$$

$$f(x_0) + 3hf'(x_0) + \frac{(3h)^2}{2!}f''(x_0) + \frac{(3h)^3}{3!}f'''(x_0) + \frac{(3h)^4}{4!}f^{4}(x_0) + \frac{(3h)^5}{5!}f^{5}(x_0) + \dots\Big]$$

$$= 3hf(x_0) + \frac{9}{2}h^2f'(x_0) + \frac{9}{2}h^3f''(x_0) + \frac{27}{8}h^4f'''(x_0) + \frac{33}{16}h^5f^4(x_0) + \dots$$

Put in (ii)

$$E = I_1 - I_2$$

$$= \frac{-3}{80}h^5 f^4(x_0) + \dots$$

$$\approx \frac{-3}{80}h^5 f^4(x_0) \qquad \text{(Required)}$$

IV Error Analysis of Bode's Rule :-

$; n = 4$

$$E = X - A$$

$$= \int_{x_0}^{x_4} f(x)\,dx - \frac{2h}{45}\Big\{7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)\Big\}$$

$$= I_1 - I_2 \longrightarrow (iii)$$

Now $I_1 = \int_{x_0}^{x_4} f(x)\,dx$

$$= F(x_4) - F(x_0) \qquad \because f(x) = F'(x)$$

$$= F(x_0 + 4h) - F(x_0)$$

$$= F(x_0) + 4hF'(x_0) + \frac{(4h)^2}{2!}F''(x_0) + \frac{(4h)^3}{3!}F'''(x_0) + \frac{(4h)^4}{4!}F^{4}(x_0) + \frac{(4h)^5}{5!}F^{5}(x_0) + \dots - F(x_0)$$

$$= 4hf(x_0) + 8h^2f'(x_0) + \frac{32}{3}h^3f''(x_0) + \frac{32h^4}{3}f'''(x_0) + \frac{128}{15}h^5f^{iv}(x_0) + \frac{256}{45}h^6f^{(v)}(x_0) + \frac{1024}{315}h^7f^{6}(x_0) + \dots$$

Now $I_2 = \frac{2h}{45}\Big\{7f(x_0) + 32f(x_0+h) + 12f(x_0+2h) + 32f(x_0+3h) + 7f(x_0+4h)\Big\}$

$$I_2 = \frac{2h}{45}\Big[7f(x_0) + 32\Big\{ f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \frac{h^3}{3!}f'''(x_0) + \frac{h^4}{4!}f^{4}(x_0) + \dots \Big\}$$

$$+ 12\Big\{ f(x_0) + 2hf'(x_0) + \frac{(2h)^2}{2!}f''(x_0) + \frac{(2h)^3}{3!}f'''(x_0) + \frac{(2h)^4}{4!}f^{4}(x_0) + \frac{(2h)^5}{5!}f^{5}(x_0) + \frac{(2h)^6}{6!}f^{6}(x_0) + \dots \Big\}$$

$$+ 32\Big\{ f(x_0) + 3hf'(x_0) + \frac{(3h)^2}{2!}f''(x_0) + \frac{(3h)^3}{3!}f'''(x_0) + \frac{(3h)^4}{4!}f^{4}(x_0) + \frac{(3h)^5}{5!}f^{5}(x_0) + \frac{(3h)^6}{6!}f^{6}(x_0) + \dots \Big\}$$

$$+ 7\Big\{ f(x_0) + 4hf'(x_0) + \frac{(4h)^2}{2!}f''(x_0) + \frac{(4h)^3}{3!}f'''(x_0) + \frac{(4h)^4}{4!}f^{4}(x_0) + \frac{(4h)^5}{5!}f^{5}(x_0) + \frac{(4h)^6}{6!}f^{6}(x_0) + \dots \Big\} \Big]$$

$$= 4hf(x_0) + 8h^2f'(x_0) + \frac{32}{3}h^3f''(x_0) + \frac{32}{3}h^4f'''(x_0) + \frac{128}{15}h^5f^{4}(x_0) + \frac{256}{45}h^6f^{5}(x_0) + \frac{88}{27}h^7f^{6}(x_0) + \frac{1552}{945}h^8f^{7}(x_0) + \dots$$

Put in (iii)

$$E = 4hf(x_0) + 8h^2f'(x_0) + \frac{32}{3}h^3f''(x_0) + \frac{32}{3}h^4f'''(x_0) + \frac{128}{15}h^5f^{4}(x_0) + \frac{256}{45}h^6f^{5}(x_0) + \frac{1024}{315}h^7f^{6}(x_0) + \frac{512}{315}h^8f^{7}(x_0) + \dots -$$

$$4hf(x_0) - 8h^2f'(x_0) - \frac{32}{3}h^3f''(x_0) - \frac{32}{3}h^4f'''(x_0) - \frac{128}{15}h^5f^{4}(x_0) - \frac{256}{45}h^6f^{5}(x_0) - \frac{88}{27}h^7f^{6}(x_0) - \dots$$

Neglecting h^8 and higher terms, we get

$$E = \frac{-8h^7f^{6}(x_0)}{945}$$

(Required)

(V) Error Anlysis of Weddle's Rule;

$n = 6$

$$E = X_{x_i} - A$$

$$= \int_{x_0}^{x_6} f(x)\,dx - \frac{3h}{10}\Big\{ f(x_0) + 5f(x_1) + f(x_2) + 6f(x_3) + f(x_4) + 5f(x_5) + f(x_6) \Big\}$$

$$= I_1 - I_2 \longrightarrow (iv)$$

$$I_1 = \int_{x_0}^{x_6} f(x)\,dx \qquad (F'(x) = f(x))$$

$$= F(x_6) - F(x_0)$$

$$= F(x_0 + 6h) - F(x_0)$$

$$= F(x_0) + 6hF'(x_0) + \frac{(6h)^2}{2!}F''(x_0) + \frac{(6h)^3}{3!}F'''(x_0) + \frac{(6h)^4}{4!}F^{4}(x_0) + \frac{(6h)^5}{5!}F^{5}(x_0) + \frac{(6h)^6}{6!}F^{6}(x_0) + \frac{(6h)^7}{7!}F^{7}(x_0) + \dots - F(x_0)$$

$$I_1 = 6hf(x_0) + 18h^2f'(x_0) + 36h^3f''(x_0) + 54h^4f'''(x_0) + \frac{324}{5}h^5f^{4}(x_0) + \frac{324}{5}h^6f^{5}(x_0) + \frac{1944}{35}h^7f^{6}(x_0) + \frac{1458}{35}h^8f^{7}(x_0) + \dots$$

$$I = \frac{3h}{10}\left[f(x_0) + 5f(x_0+h) + f(x_0+2h) + 6f(x_0+3h) + f(x_0+4h) + 5f(x_0+5h) + f(x_0+6h) \right] \longrightarrow *$$

By Taylor's Series find $f(x_0+nh)$

$n = 1, 2, 3, 4, 5, 6$

then put in $*$ we get

$$= \frac{3h}{10}\Big\{ 20f(x_0) + 60hf'(x_0) + 120h^2f''(x_0) + 180f'''(x_0)h^3 + 216h^4f^4(x_0) + 216h^5f^5(x_0)$$
$$+ \frac{111}{6}h^6f^6(x_0) + \frac{1945}{7}h^7f^7(x_0) + \frac{77873}{840}h^8f^8(x_0) + \cdots\cdots \Big\}$$

$$= 6hf(x_0) + 18h^2f'(x_0) + 36h^3f''(x_0) + 54h^4f'''(x_0) + \frac{324}{5}h^5f^4(x_0) + \frac{324}{5}h^6f^5(x_0)$$
$$+ \frac{1111}{20}h^7f^6(x_0) + \frac{1167}{34}h^8f^7(x_0) + \cdots\cdots$$

put in (iv)

$$E = \cancel{6hf(x_0)} + \cancel{18h^2f'(x_0)} + \cancel{36h^3f''(x_0)} + \cancel{54h^4f'''(x_0)} + \cancel{\frac{324}{5}h^5f^4(x_0)} + \cancel{\frac{324}{5}}h^6f^5(x_0) +$$
$$\frac{1944}{35}h^7f^6(x_0) + \frac{1458}{35}h^8f^7(x_0) + \cdots\cdots$$
$$- \cancel{6hf(x_0)} - \cancel{18h^2f'(x_0)} - \cancel{36h^3f''(x_0)} -$$

$$\cancel{54h^4f'''(x_0)} - \cancel{\frac{324}{5}h^5f^4(x_0)} - \frac{324}{5}h^6f^5(x_0) -$$
$$\frac{1111}{20}h^7f^6(x_0) - \frac{1167}{34}h^8f^7(x_0) - \cdots\cdots$$

$$= \frac{-1}{140}h^7f^{vi}(x_0) + \frac{8727}{1190}h^8f^{vii}(x_0) + \cdots\cdots$$

$$= \frac{-1}{140}h^7f^6(x_0)$$

by neglecting h^8 & higher power.

which is required error.

DIFFERENCE EQUATION

A difference equation is a relation between the differences of an unknown function at one or more general values of the arguments.

For example:

$$y_{n+2} + 2y_{n+1} + y_n = n^2$$
$$\Delta^2 y_n + \Delta 2y_n + y_n = n^2$$
$$(E^2 + 2E + 1)y_n = n^2$$

* **Order of Difference Equation:**

The order of difference equation is the difference between largest and smallest argument (suffix) in equation

For example:

$$y_{k+2} + 2y_{k+1} + y_k = k^2$$

order is $k+2-k = 2$.

* **FORMATION OF DIFFERENCE EQUATION :-**

The following example illustrates the way in which differense equation is formed.

Example (i)

From $y_n = A2^n + B(-3)^n$. Derive difference equation not containing A and B.

Solution:-

$$y_n = A2^n + B(-3)^n \longrightarrow (i)$$
$$y_{n+1} = A(2)^{n+1} + B(-3)^{n+1} \longrightarrow (ii)$$
$$y_{n+2} = A(2)^{n+2} + B(-3)^{n+2} \longrightarrow (iii)$$

From (i), (ii) and (iii)

$$y_n = A2^n + B(-3)^n$$
$$y_{n+1} = 2A2^n + (-3)B(-3)^n$$
$$y_{n+2} = 4A2^n + 9B(-3)^n$$

Eliminate $A2^n$ and $B(-3)^n$ from above equations.

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 2 & -3 \\ y_{n+2} & 4 & 9 \end{vmatrix} = 0$$

$$y_n[18+12] - 1[9y_{n+1} + 3y_{n+2}] + 1[4y_{n+1} - 2y_{n+2}]$$

$$= y_{n+2} + y_{n+1} - 6y_n = 0$$

which is required difference equation.

Example (ii)

Eliminate constant from the equation $y_n = A_1 + (-1)^n A_2 + A_3 n$ and derive corresponding difference equation.

Solution :-

$$y_n = A_1 + (-1)^n A_2 + A_3 n \longrightarrow (i)$$
$$y_{n+1} = A_1 + (-1)^{n+1} A_2 + A_3(n+1)$$
$$y_{n+1} = A_1 + (-1)(-1)^n A_2 + A_3 n + A_3 \longrightarrow (ii)$$
$$y_{n+2} = A_1 + (-1)^{n+2} A_2 + A_3(n+2)$$
$$y_{n+2} = A_1 + (-1)^n A_2 + n A_3 + 2A_3 \longrightarrow (iii)$$
$$y_{n+3} = A_1 + (-1)^3(-1)^n A_2 + n A_3 + 3A_3 \longrightarrow (iv)$$

Subtract (i) from (iii)

$$y_{n+2} - y_n = 2A_3 \longrightarrow (v)$$

Subtract (ii) from (iv)

$$y_{n+3} - y_{n+1} = 2A_3 \longrightarrow (vi)$$

From (v) and (vi)

$$y_{n+3} - y_{n+1} = y_{n+2} - y_n$$
$$\Rightarrow y_{n+3} - y_{n+2} - y_{n+1} + y_n = 0$$

is required difference equation. (Ans).

Example (iii):

Also for $y_n = (A+Bn)3^n$ the difference equation is

$$y_{n+2} - 6y_{n+1} + 9y_n = 0. \quad (Ans)$$

Linear Difference Equations:

LDE is that in which y_n, y_{n+1}, y_{n+2} etc occur to first degree only and are not multiplied.

A Linear difference equation with constant coefficient is of the form
$$a_0 y_{n+r} + a_1 y_{n+r-1} + a_2 y_{n+r-2} + \dots + a_r y_n = \phi(n)$$

The equation can be written symbolically as
$$f(E)y_n = \phi(n)$$
If $\phi(n) = 0$ homogenuous $\longrightarrow$ (i)
If $\phi(n) \neq 0$ Non homogenuous $\longrightarrow$ (ii)

1^{st} we Solve associated homogenous
$$f(E)y_n = 0$$
The solution of that equation is called complementry function (C.F). It contains arbitrary constant equal to order of that equation.

2^{nd} Solve associated Non-homogenous
$$f(E)y_n = \phi(n)$$
The solution is called perticular Solution. which doesnot contain arbitrary constant.

General solution of $f(E)y_n = \phi(n)$ is
$$y_n = C.F + P.S.$$

Rule for finding Complementary Function (C.F) of equation $f(E)y_n = \phi(n)$.

Solve $f(E) = 0$ which is called Auxiliary equation.

Let $\alpha_1, \alpha_2, \alpha_3, \ldots$ be root of auxiliary equation

(I) If $\alpha_1, \alpha_2, \ldots\ldots$ are all real and distinct

Then C.F is

$$y_n = C_1\alpha_1^n + C_2\alpha_2^n + C_3\alpha_3^n + \ldots\ldots$$

(II) If two roots are equal $\alpha_1 = \alpha_2 = \alpha$

Then C.F is

$$y_n = (C_1 + C_2 n)\alpha^n + C_3\alpha_3^n + \ldots\ldots$$

Similarly if three roots are equal

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha$$

$$y_n = (C_1 + C_2 n + C_3 n^2)\alpha^n + C_4\alpha_4^n + \ldots\ldots$$

(III) If two roots are imaginary say $\alpha \pm i\beta$ then C.F is

$$y_n = R^n(C_1\cos n\theta + C_2 \sin n\theta) + C_3\alpha_3^n + \ldots\ldots$$

where

$$R = \sqrt{\alpha^2 + \beta^2}$$

$$\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

Questions :- Solve

(i) $y_{k+2} - 3y_{k+1} + 2y_k = 0$

Equation in E-operator form is

$$E^2 y_k - 3E y_k + 2y_k = 0$$

$$(E^2 - 3E + 2)y_k = 0$$

The auxiliary equation is

$$E^2 - 3E + 2 = 0$$

$$(E-1)(E-2) = 0$$

$$\Rightarrow E = 1, 2$$

$$y_k = C_1(1)^n + C_2(2)^n \quad \text{(Answer)}$$

(ii) $U_{n+3} - 2U_{n+2} - 5U_{n+1} + 6U_n = 0$

$$E^3 U_n - 2E^2 U_n - 5E U_n + 6U_n = 0$$

$$(E^3 - 2E^2 - 5E + 6)U_n = 0$$

Auxiliary equation is

$$E^3 - 2E^2 - 5E + 6 = 0$$

$$(E-1)(E^2 - E - 6) = 0$$

$$(E-1)(E+2)(E-3) = 0$$

$$\Rightarrow E = 1, -2, +3$$

$$U_n = C_1(1)^n + C_2(-2)^n + C_3(+3)^n$$

(iii) Find C.F of $(E^2 + 2E + 4)y_k = 0$.

$$E^2 + 2E + 4 = 0$$

$$E = \frac{-2 \pm \sqrt{4 - 4(1)(4)}}{2}$$

$$= \frac{-2 \pm \sqrt{12}\, i}{2}$$

$$= -1 \pm \sqrt{3}\, i$$

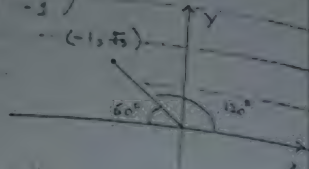
$a = -1, \quad b = +\sqrt{3}$

$R = \sqrt{(-1)^2 + (\sqrt{3})^2}$

$= \sqrt{1+3}$

$= \sqrt{4} = 2$

$\theta_1 = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{+\sqrt{3}}{-1}\right)$



$\theta_1 = 120^\circ = \frac{2\pi}{3}$

Solution is

$y_k = R^k (c_1 \cos k\theta_1 + c_2 \sin k\theta_1)$

$= 2^k \left(c_1 \cos \frac{2\pi}{3} k + c_2 \sin \frac{2\pi}{3} k\right).$

(Ans).

(Exercise)

Solve following difference equations:-

(i) $U_{n+2} - 2U_{n+1} + U_n = 0$

Ans: $U_n = c_1 + c_2 n$

(ii) $y_{n+3} - 3y_{n+2} + 4y_n = 0$

Ans: $y_n = c_1(-1)^n + (c_2 + c_3 n) 2^n$

(iii) $(E^2 + 2E + 2) f_n = 0$

Ans: $2^{n/2} \left[\cos \frac{3\pi n}{4} + B \sin \frac{3\pi}{4} n\right].$

(iv) $y_{k+4} - 2y_{k+3} + 2y_{k+2} + (-2) y_{k+1} + y_k = 0$

Ans:

$y_k = (A + Bk) + \left(C \cos \frac{\pi}{2} k + D \sin \frac{\pi}{2} k\right)$

(v) $y_{n+2} - y_{n+1} + y_n = 0 \quad ; \quad y_0 = 1$

$y_1 = 1 + \frac{\sqrt{3}}{2}$

Ans: $y_n = \cos \frac{\pi}{3} n + \sin \frac{\pi}{3} n.$

(vi) $y_{k+3} + 6y_{k+2} + 11y_{k+1} + 6y_k = 0$

Ans: $y_k = c_1(-1)^k + c_2(-2)^k + c_3(-3)^k$

(vii) $y_{t+2} + \frac{1}{4} y_t = 0 \quad ; \quad y_0 = 1, \; y_1 = 2$

Ans: $y_t = \left(\frac{1}{2}\right)^t \left(\cos \frac{\pi}{2} t + 2 \sin \frac{\pi}{2} t\right).$

(viii) $\Delta^3 y_n - 5\Delta y_n + 4y_n = 0$

Ans: $y_n = c_1 2^n + c_2 \left(\frac{1 + \sqrt{17}}{2}\right)^n + c_3 \left(\frac{1 - \sqrt{17}}{2}\right)^n.$

(ix) $y_{m+2} + 16y_{m-1} = 0$

Ans:

$$y_m = 2^m\left[c_1\cos\frac{\pi}{4}m + c_2\sin\frac{\pi}{4}m + c_3\cos\frac{3\pi}{4}m + c_4\sin 3\pi/4\, m\right]$$

(x) Let u_k and v_k be Solution of

$\rightarrow$ * $y_{k+2} + a_1y_{k+1} + a_2y_k = 0$ then show that $c_1u_k + c_2v_k$ is also solution.

Sol: u_k and v_k be solution then

$$u_{k+2} + a_1u_{k+1} + a_2u_k = 0 \rightarrow (i)$$

$$v_{k+2} + a_1v_{k+1} + a_2v_k = 0 \rightarrow (ii)$$

we show that $c_1u_k + c_2v_k$ is also solution.

xing (i) by c_1 and c_2 by (ii) then add

$$(c_1u_{k+2} + c_2v_{k+2}) + a_1(c_1u_{k+1} + c_2v_{k+1}) + a_2(c_1u_k + c_2v_k) = 0.$$

Comparing with * it follows that $c_1u_k + c_2v_k$ is also solution.

Solution Of Non-homogenuous Linear eq. $f(E)y_n = \phi(n)$.

Type (1):

When $\phi(n) = $ Constant

The trail substitution for perticulee solution is

$$y_n = c$$

(i) If sum of coeff. of Auxilianly equation is zero then $y_n = c$ fails.

In this case $y_n = c$ (some power of n).

(ii) If sum of coeff. of Auxiliarly equation is zero then '1' be root of equation.

(iii) If 1 is single root put $y_n = cn$

If 1 is double root put $y_n = cn^2$

If 1 is triple root put $y_n = cn^3$.

.... So on.

* Question: Solve

$$y_{k+3} + 3y_{k+2} - 2y_k = 5 \longrightarrow (i)$$

Solution: $E^3y_k + 3E^2y_k - 2y_k = 5$

$$(E^3 + 3E^2 - 2)y_k = 5$$

Auxiliarly equation is

$$E^3 + 3E^2 - 2 = 0$$

$E = -1 \Rightarrow E^3 + 3E^2 - 2 = 0$

$$(-1)^3 + 3(-1)^2 - 2 = 0.$$

$$-1 + 3 - 2 = 0$$

So $E = -1$ is root.

$$\begin{array}{r|rrrr} -1 & 1 & 3 & 0 & -2 \\ & & -1 & -2 & +2 \\ \hline & 1 & 2 & -2 & \boxed{0} \end{array}$$

$\Rightarrow E^2 + 2E - 2 = 0$

$\Rightarrow E = -1 \pm \sqrt{3}$

$\Rightarrow$ Roots are -1, $-1 \pm \sqrt{3}$

Complement function is

$$y_k = A(-1)^k + B(-1+\sqrt{3})^k + C(-1-\sqrt{3})^k$$

and particular Solution put $y_k = C$

$y_k = C$

$y_{k+1} = C$

$y_{k+2} = C$

$y_{k+3} = C$

put in (i)

$C + 3C - 2C = 5$

$2C = 5$

$C = \frac{5}{2}$

$y_k = 5/2$

P.S $= y_k = 5/2$

General sol $=$ C.F $+$ P.S

$$= A(-1)^k + B(-1+\sqrt{3})^k + C(-1-\sqrt{3})^k + 5/2$$

(Answer).

Question: $y_{t+2} + \frac{1}{4} y_t = 2$; $y_0 = y_1 = 1$

Solution: $y_{t+2} + \frac{1}{4} y_t = 2$

$\Rightarrow E^2 y_t + 1/4\, y_t = 2$

$\Rightarrow (E^2 + 1/4)\, y_t = 2$

Ch. eq is $E^2 + \frac{1}{4} = 0 \Rightarrow E = \pm \frac{1}{2} i$

C.F equation is $y_t = R^t [A \cos t\theta + B \sin t\theta]$

where $R = \sqrt{a^2 + b^2} = \sqrt{0 + (1/2)^2} = \frac{1}{2}$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{1/2}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$$

$$y_t = \left(\frac{1}{2}\right)^t \left[A \cos \frac{\pi}{2} t + B \sin \frac{\pi}{2} t\right]$$

For P.S put $y_t = C$

$y_{t+1} = y_{t+2} = C$.

then $C + \frac{1}{4} C = 2$

$\frac{5}{4} C = 2 \Rightarrow C = \frac{8}{5}$.

$y_t = \frac{8}{5}$

G.S $=$ C.F $+$ P.S

$$= \left(\frac{1}{2}\right)^t \left[A \cos \frac{\pi t}{2} + B \sin \frac{\pi}{2} t\right] + \frac{8}{5}.$$

Also find A and B Give $y_0 = y_1 = 1$

When $t = 0$

$$y_0 = \left(\frac{1}{2}\right)^0 [A\cos(0) + B\sin(0)] + 8/5$$

$$1 = A + \frac{8}{5}$$

$$\Rightarrow A = -3/5$$

When $t = 1$

$$y_1 = \left(\frac{1}{2}\right)^1 \left[A\cos\left(\frac{\pi}{2}\right) + B\sin\left(\frac{\pi}{2}\right)\right] + \frac{8}{5}$$

$$1 = \frac{1}{2}\left[\frac{-3}{5}(0) + B(1)\right] + \frac{8}{5}$$

$$1 = \frac{B}{2} + \frac{8}{5}$$

$$\Rightarrow B = 2\left(1 - \frac{8}{5}\right) = \frac{-6}{5}$$

Then

$$y_t = \left(\frac{1}{2}\right)^t \left[\frac{-3}{5}\cos\frac{\pi}{2}t - \frac{6}{5}\sin\frac{\pi}{2}t\right] + \frac{8}{5}$$

(Answer).

(Exercise)

Solve

1) $y_{n+2} - 4y_{n+1} + y_n = 1$

Ans: $y_n = c_1(2+\sqrt{3})^n + c_2(2-\sqrt{3})^n - \frac{1}{2}$

2) $y_{n+2} - 3y_{n+1} - 4y_n = 6$

Ans: $y_n = c_1(-1)^n + c_2 4^n + 1$

3) $u_{n+2} - 4u_{n+1} + u_n = 3$

Ans: $u_n = c_1(2+\sqrt{3})^n + c_2(2-\sqrt{3})^n - \frac{3}{2}$

4) $y_{n+3} + 6y_{n+2} + 11y_{n+1} = 5$

5) $\Delta^2 y_k + 3\Delta y_k = 9$; $y_0 = 1$, $y_1 = 0$

6) $(E^3 - 4E^2 + 2E + 1)y_n = 14$

Ans: $y_n = c_1(+1)^n + c_2\left(\frac{3+\sqrt{13}}{2}\right)^n +$

$c_3\left(\frac{3-\sqrt{13}}{2}\right)^n - \frac{14}{3}n$

Type (2):

When $\phi(n) = \alpha a^n$

where α is constant.

For P.S trail substitution is

$y_n = C a^n$ where a is not root.

when a is root of auxiliary equation, trail substitution is given below.

(i) a is single root put $y_n = C n a^n$

(ii) a is double root put $y_n = C n^2 a^n$

(iii) a is triple root put $y_n = C n^3 a^n$

.... so on.

Example :-

Solve $U_{n+2} - 7U_{n+1} + 10U_n = 12(4)^n \rightarrow (i)$

Solution :- $U_{n+2} - 7U_{n+1} + 10U_n = 12(4)^n$

$E^2 U_n - 7E U_n + 10U_n = 12(4)^n$

$(E^2 - 7E + 10)U_n = 12(4)^n$ For C.F

Ch. equation $E^2 - 7E + 10 = 0$

$E^2 - 2E - 5E + 10 = 0$

$E(E-2) - 5(E-2) = 0$

$(E-2)(E-5) = 0$

$\Rightarrow E = +2, 5$

C.F is $U_n = A(2)^n + B(5)^n$

For P.S

put $U_n = C4^n$ Here $a = 4$

$U_{n+1} = C(4)^{n+1}$

$U_{n+2} = C(4)^{n+2}$

Put in (i)

$C4^{n+2} - 7C4^{n+1} + 10C4^n = 12\,4^n$

$4^n[C(4)^2 - 7C(4)^1 + 10C] = 12\,4^n$

$\Rightarrow 16C - 28C + 10C = 12$

$\Rightarrow -2C = 12$

$C = -6$

$\Rightarrow U_n = C4^n$

$= -6(4)^n$

General solution = C.F + P.S

$U_n = A(2)^n + B(5)^n - 6\,4^n$

(Answer)

Example:

Solve

$y_{k+2} - 4y_{k+1} + 4y_k = 2^{k+1} \rightarrow (i)$

Solution :

$y_{k+2} - 4y_{k+1} + 4y_k = 2 \cdot 2^k$

$E^2 y_k - 4E y_k + 4y_k = 2(2)^k$

$\Rightarrow (E^2 - 4E + 4)y_k = 2(2)^k$.

For C.F

Ch. eq. $E^2 - 4E + 4 = 0$

$E^2 - 2E - 2E + 4 = 0$

$E(E-2) - 2(E-2) = 0$

$\Rightarrow (E-2)(E-2) = 0$

$\Rightarrow E = 2, 2$.

$y_k = (A + Bk)(2)^k$

For Particular Solution.

put $y_k^* = Ck^2 2^k$

($\because$ 2 is double root)

$y_{k+1} = C(k+1)^2 (2)^{k+1}$

$y_{k+2} = C(k+2)^2 (2)^{k+2}$

Put in (i)

$C(k+2)^2 2^{k+2} - 4C(k+1)^2 2^{k+1} + 4Ck^2 2^k = 2.2^k$

$\Rightarrow [C(k^2+4+4k)2^2 - 4C(k^2+1+2k)2^1 + 4Ck^2]2^k = 2.2^k$

$\Rightarrow C[(k^2+4+4k)4 - 4(2)(k^2+1+2k) + 4k^2] = 2$

$\Rightarrow C[\cancel{4k^2} + 16 + \cancel{16k} - \cancel{8k^2} - 8 - \cancel{16k} + \cancel{4k^2}] = 2$

$\Rightarrow C[8] = 2$

$\Rightarrow C = \frac{1}{4}$

$y_k^* = \frac{1}{4} k^2 2^k$

General solution is

$C.F + P.S$

$\Rightarrow y_k = y_k + y_k^*$

$= (A+Bk)2^k + \frac{1}{4} k^2 2^k$

(Answer)

Example: Solve $U_{n+2} - 7U_{n+1} + 10U_n = 12.5^n$

Solution:

$U_{n+2} - 7U_{n+1} + 10U_n = 12\,5^n$

$E^2 U_n - 7EU_n + 10U_n = 12(5)^n$

$(E^2 - 7E + 10)U_n = 12(5)^n$

For C.F

ch.eq. $E^2 - 7E + 10 = 0$

$E = 2, 5 \Rightarrow U_n = C_1 2^n + C_2 5^n$

For P.S put $U_n^* = Cn5^n$

$U_{n+1} = C(n+1)5^{n+1}$

$U_{n+2} = C(n+2)5^{n+2}$

then $U_{n+2} - 7U_{n+1} + 10U_n = 2.5^n$ becomes

$C(n+2)5^{n+2} - 7C(n+1)5^{n+1} + 10Cn5^n = 2.5^n$

$5^n[C(n+2)5^2 - 7C(n+1)5 + 10Cn] = 2.5^n$

$\Rightarrow C(n+2)5^2 - 35C(n+1) + 10C(n) = 2$

$C[25n + 50 - 35n - 35 + 10n] = 2$

$C[25n + 50 - 35n - 35] = 2$

$C(15) = 12$

$C = \frac{12}{15} = \frac{4}{5}$

$\Rightarrow U_n^* = \frac{4}{5} n 5^n$

General Sol $= C.F + P.S$

$= U_n + U_n^*$

$= C_1 2^n + C_2 5^n + \frac{4}{5} n 5^n$

(Answer)

EXERCISE

<u>Solve:</u>

1) $y_{n+4} - A^4 y_n = 2^n$

Ans: $y_n = c_1 A^n + c_2 (-A)^n + A^n \left(c_3 \cos \frac{\pi}{2} n + c_4 \sin \frac{\pi}{2} n \right) + \frac{1}{16 - A^4} 2^n$

2) $y_{n+2} + 4y_{n+1} - 4y_n = 2^{n+2}$; $y_0 = 1$, $y_1 = 2$.

Ans: $y_n = \left(1 - \frac{n}{2}\right) 2^n + \frac{1}{2} n^2 2^n$

3) $y_{k+2} - 6y_{k+1} + 8y_k = 2 \cdot 3^k$

Ans: $y_k = C_1 2^k + C_2 4^k - 2 \cdot 3^k$

4) $y_{k+3} - y_{k+2} + y_{k+1} - y_k = 3^k$

Ans: $y_n = C_1 + C_2 \cos \frac{n\pi}{2} + C_3 \sin \frac{\pi n}{2} + \frac{1}{2} 3^k$

5) $y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n = 3^n$

Ans: $y_n = C_1 + nC_2 + n^2 C_3 + 3^n$

6) $y_{n+2} - 3y_{n+1} + 2y_n = 7 \cdot 2^n$

Ans: $y_n = C_1 + C_2 n^2 + 7n2^{n-1}$

Type (3):

When $\phi(n)$ = polynomial

Like $\phi(n) = 1 + k^2$ or $k^3 + k^2 + k + 1$

The trail substitution for P.S is

$y_n = B_0 + B_1 n + B_2 n^2 + \ldots\ldots B_k n^k$ (degree of polyn)

(i) If sum of coeffs. in $f(E) y_n = 0$ equation is zero then 1 is root of above equation

then $y_n = B_0 + B_1 n + B_2 n^2 + \ldots\ldots + B_k n^k$ fails.

In this case multiply trail solution as below

(1) 1 is single put $y_n = n(B_0 + B_1 n + \ldots\ldots B_k n^k)$

(2) 1 is double put $y_n = n^2(\quad " \quad " \quad)$

(3) 1 is triple put $y_n = n^3(\quad " \quad " \quad)$

...... so on.

<u>Example:</u> Solve

$y_{n+2} - 2y_{n+1} + y_n = n^2$

<u>Solution:</u> $y_{n+2} - 2y_{n+1} + y_n = n^2$

$E^2 y_n - 2Ey_n + y_n = n^2$

$\Rightarrow (E^2 - 2E + 1) y_n = n^2$

For Complementry function ...

Ch. eq. $E^2 - 2E + 1 = 0$

$\Rightarrow (E-1)^2 = 0$

$\Rightarrow E = 1, 1$

C.F $\Rightarrow y_n = (c_1 + c_2 n)(1)^n$

$y_n = c_1 + c_2 n.$

As 1 is double root so put for P.S

$y_n^* = n^2 (B_0 + B_1 n + B_2 n^2)$

$y_n = n^2 B_0 + B_1 n^3 + B_2 n^4$

$y_{n+1} = (n+1)^2 B_0 + (n+1)^3 B_1 + (n+1)^4 B_2$

$y_{n+2} = (n+2)^2 B_0 + (n+2)^3 B_1 + (n+2)^4 B_2$

then $y_{n+2} - 2y_{n+1} + y_n = n^2$ becomes

$B_0(n+2)^2 + B_1(n+2)^3 + B_2(n+2)^4 +$
$(-2)[B_0(n+1)^2 + B_1(n+1)^3 + B_2(n+1)^4] +$
$B_0 n^2 + B_1 n^3 + B_2 n^4 = n^2$

$\Rightarrow B_0(n^2+4+4n) + B_1(n^3+6n^2+12n+8) +$
$B_2(n^4+8n^3+24n^2+32n+16)$
$-2B_0(n^2+2n+1) - 2B_1(n^3+3n^2+3n+1)$
$-2B_2(n^4+4n^3+6n^2+4n+1) +$
$B_0 n^2 + B_1 n^3 + B_2 n^4 = n^2$

Equating coeff. of n^4:

$B_2 - 2B_2 + B_2 = 0$

Not gives information

Equating coeff of n^3:

$B_1 + 8B_2 - 2B_1 - 8B_2 + B_1 = 0$

$0 = 0$

Equating Coeff. of n^2:

$\cancel{B_0} + \cancel{6B_1} + 24B_2 - \cancel{2B_0} - \cancel{6B_1} - 12B_2 + \cancel{B_0} = 1$

$12B_2 = 1$

$B_2 = \frac{1}{12}.$

Equating Coeff. of n:

$4B_0 + 12B_1 + 32B_2 - 4B_0 - 6B_1 - 8B_2 = 0$

$6B_1 + 24B_2 = 0$

$B_1 = -4B_2$

$B_1 = -4\left(\frac{1}{12}\right) = \frac{-1}{3}$

Equating constant term:

$4B_0 + 8B_1 + 16B_2 - 2B_0 - 2B_1 - 2B_2 = 0$

$2B_0 + 6B_1 + 14B_2 = 0$

$B_0 + 3B_1 + 7B_2 = 0$

$B_0 + 3\left(\frac{-1}{3}\right) + 7\left(\frac{1}{12}\right) = 0$

$B_0 - 1 + \frac{7}{12} = 0$

$\Rightarrow B_0 = 5/12.$

So $y_n^* = n^2\left(\frac{5}{12} + \frac{-1}{3}n + \frac{1}{12}n^2\right)$

$= \frac{5}{12}n^2 - \frac{1}{3}n^3 + \frac{1}{12}n^4.$

General solution is
$= C.F + P.S$

$$y_n = C_1 + C_2 n + \frac{5}{12} n^2 - \frac{1}{3} n^3 + \frac{1}{12} n^4$$

(Ans)

Example:

Solve $\Delta^2 y_k + 3\Delta y_k + 3 y_k = k^2 + 1 \rightarrow ⓐ$

Solution: $\Delta^2 y_k + 3\Delta y_k + 3 y_k = k^2 + 1$

As $\Delta y_k = y_{k+1} - y_k$

So $\Delta(\Delta y_k) + 3\Delta y_k + 3 y_k = k^2 - 1$

$\Delta(y_{k+1} - y_k) + 3(y_{k+1} - y_k) + 3 y_k = k^2 - 1$

$\Delta y_{k+1} - \Delta y_k + 3 y_{k+1} - \cancel{3} y_k + 3 y_k = k^2 - 1$

$y_{k+2} - y_{k+1} - y_{k+1} + y_k + 3 y_{k+1} = k^2 + 1$

$y_{k+2} - 1 y_{k+1} + y_k = k^2 + 1 \longrightarrow ①$

For Complementry function:

Ch equation $y_{k+2} - y_{k+1} + y_k = 0$

$E^2 y_k - E y_k + y_k = 0$

$(E^2 - E + 1) y_k = 0, \quad y_k \neq 0$

$\Rightarrow \quad E^2 - E + 1 = 0.$

$$E = \frac{1 \pm \sqrt{1 - 4(1)(1)}}{2}$$

$$E = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm \sqrt{3}\, i}{2}$$

As

$$y_k = R^k \left[A_1 \cos k\theta + A_2 \sin k\theta \right]$$

$$R = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{\frac{4}{4}} = 1$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right)$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$$

$$y_k = (1)^k \left[A_1 \cos \frac{\pi}{3} k + A_2 \sin \frac{\pi}{3} k \right] \longrightarrow *$$

For Particular Solution

Put $y_k^* = (B_0 + B_1 K + B_2 K^2)$

then ① becomes

$B_0 + B_1(k+2) + B_2(k+2)^2 - B_0 - B_1(k+1) - B_2(k+1)^2$
$+ B_0 + B_1 k + B_2 K^2 = K^2 + 1$

$B_0 + B_1(k+2) + B_2(K^2 + 4k + 4) - B_0 - B_1(k+1)$
$- B_2(k^2 + 2k + 1) + B_0 + B_1 k + B_2 K^2$
$= K^2 + 1.$

Equating Coeff. of K^2:

$B_2 - \cancel{B_2} + \cancel{B_2} = 1$

$\Rightarrow B_2 = 1$

Equating Coeff of k:

$B_1 + 4B_2 - \cancel{B_1} - 2B_2 + \cancel{B_1} = 0$

$B_1 + 2B_2 = 0$

$B_1 + 2(1) = 0$

$\Rightarrow \quad B_1 = -2$

Equating Constant term :-

$B_0 + 2B_1 + 4B_2 - \cancel{B_0} - B_1 - B_2 + B_0 = 1$

$B_0 + B_1 + 3B_2 = 1$

$B_0 + (-2) + 3(1) = 1$

$B_0 + 1 = 1$

$B_0 = 0$

So y_k^* becomes

$y_k^* = B_0 + B_1 k + B_2 k^2$

$y_k^* = -2k + k^2 \longrightarrow \star'$

General Solution = C.F + P.S

$y_k' = y_k + y_k^*$

From $\star$ and $\star'$

$$y_k' = (1)^k \left[A_1 \cos \frac{\pi}{3} k + A_2 \cos \frac{\pi}{3} k \right] + -2k + k^2$$

(Ans)

Example :

Solve

$$y_{k+4} - 2y_{k+3} + 2y_{k+2} - 2y_{k+1} + y_k = K^2 \longrightarrow (1)$$

Solution :

$y_{k+4} - 2y_{k+3} + 2y_{k+2} - 2y_{k+1} + y_k = K^2$

$E^4 y_k - 2E^3 y_k + 2E^2 y_k - 2E y_k + y_k = K^2$

$(E^4 - 2E^3 + 2E^2 - 2E + 1) y_k = K^2$

For C.F

$E^4 - 2E^3 + 2E^2 - 2E + 1 = 0$

by Synthetic Division

1	1	-2	2	-2	1
		1	-1	+1	-1
1	1	-1	1	-1	0
		1	0	1	
	1	0	1	0	

$(E-1)(E-1)(E^2+1) = 0$

$\Rightarrow E = 1, 1$

$E = \sqrt{-1}$

$\Rightarrow E = \pm i$

So roots are

$E = 1, 1, \pm i$

Complementry Solution is

$$y_k = (A + Bk)(1)^k + R^k \left[A_3 \cos k\theta + A_4 \sin k\theta \right]$$

$R = \sqrt{0 + (1)^2} = 1$

$\theta = \tan^{-1}\left(\frac{1}{0}\right) = \frac{\pi}{2}$

So

$$y_k = (A + Bk)(1)^k + (1)^k \left[A_3 \cos \frac{\pi}{2} k + A_4 \sin \frac{\pi}{2} k \right] \longrightarrow \star\star$$

So P.S put $y_k = k^2(B_0 + B_1 k + B_2 k^2)$ in (i)

$$(k+4)^2[B_0 + B_1(k+4) + B_2(k+4)^2]$$
$$-2(k+3)^2[B_0 + B_1(k+3) + B_2(k+3)^2]$$
$$+2(k+2)^2[B_0 + B_1(k+2) + B_2(k+2)^2]$$
$$-2(k+1)^2[B_0 + B_1(k+1) + B_2(k+1)^2]$$
$$+k^2[B_0 + B_1 k + B_2 k^2] = k^2$$

Comparing co-efficients of like power of n on both sides

$$B_0 + 12B_1 + 9B_2 - 2B_1 - 18B_1 - 168B_2 + 2B_0 + 12B_1 + 48B_2 - 2B_0 - 6B_1 - 12B_2 + B_0 = 1 \rightarrow \star$$

$$8B_0 + 96B_1 + 256B_2 - 12B_0 - 54B_1 - 216B_2 + 8B_0 + 24B_1 + 16B_2 - 4B_0 - 6B_1 - 8B_2 = 0 \rightarrow \star'$$

$$16B_0 + 64B_1 + 256B_2 - 18B_0 - 54B_1 - 162B_2 + 8B_0 + 16B_1 + 32B_2 - 2B_0 - 2B_1 - 2B_2 = 0 \rightarrow \star''$$

From $\star$, $\star'$ and $\star''$ we have

$$B_1 = \frac{4}{315}$$

$$B_0 = \frac{131}{315}$$

and $B_2 = -\dfrac{1}{63}$

So particular solution becomes

$$y_k = k^2\left(\frac{131}{315} + \frac{4}{315}k - \frac{1}{63}k^2\right) \rightarrow \star\star'$$

From $\star\star$ and $\star\star'$ general solution is

$$y_k = (A_1 + Bk)(1)^k + \left[A_3 \cos\frac{\pi}{2}k + A_4 \sin\frac{\pi}{2}k\right](1)^k + k^2\left(\frac{131}{315} + \frac{4}{315}k - \frac{1}{63}k^2\right)$$

(Answer)

<u>Exercise :</u>

i) $\Delta^2 y_n - 7\Delta y_n - 60 y_n = 2^n + n^2 + 1$

Ans : $C_1(-4)^n + C_2(13)^n - \dfrac{1}{66}2^n - \dfrac{1}{60}\left(n^2 - \dfrac{7}{30}n + \dfrac{1699}{1800}\right)$

2) $y_{n+2} - 7y_{n+1} + 12 y_n = 12n + 8$

Ans: $y_n = 32(3)^n - 22(4)^n + 3 + 2n$

3) $\Delta^4 y_n = n$

Ans: $y_n = C_1 + C_2 n + C_3 n^2 + C_4 n^3 - \dfrac{1}{12}n^4 + \dfrac{1}{120}n^5$

4) $y_{k+2} - 2y_{k+1} + y_k = k + 1$

Ans : $y_k = (C_1 + C_2 k) + \dfrac{1}{6}k^3$

5) $y_{n+2} - 5y_{n+1} + 6y_n = 2n + 1 + 2^n$

Ans: $y_n = (C_1 - \frac{1}{2}n)2^n + C_2 3^n + n + 2$

6) $\Delta^2 y_k + 3\Delta y_k + 3 y_k = k^2 + 1$

Ans: $y_k = C_1 \cos\dfrac{k\pi}{3} + C_2 \sin\dfrac{k\pi}{3} + k^2 - 2k + 2$

Type (4):

When $\phi(n) = a^n F(n)$
where $F(n)$ is polynomial of degree t.

For P.S trail substitution
$$y_n = a^n (B_0 + B_1 n + B_2 n^2 + \dots + B_t n^t)$$

If 'a' is root of auxiliary equation then above substitution fails.
In this case multiply the trail function by suitable power of n.

If 'a' is single root then multiply by n
If a is double root then multiply by n^2
If a is triple root then multiply by n^3
..... so on.

Example: Solve
$$y_{n+2} - 2y_{n+1} + y_n = 2^n \cdot n^2 \rightarrow (i)$$

Solution: $y_{n+2} - 2y_{n+1} + y_n = 2^n \cdot n^2$
$$E^2 y_n - 2E y_n + y_n = 2^n \cdot n^2$$
$$(E^2 - 2E + 1) y_n = 2^n \cdot n^2$$
Auxiliary equation is
$$E^2 - 2E + 1 = 0$$
$$(E-1)^2 = 0$$
$$\Rightarrow E = 1, 1$$

So C.F is $y_n = (C_1 + C_2 n)(1)^n$
$$y_n = C_1 + C_2 n$$

As for P.S put $y_n = 2^n (B_0 + B_1 n + B_2 n^2)$ in (i) we have

$$\Rightarrow 2^{(n+2)}(B_0 + B_1(n+2) + B_2(n+2)^2) - 2\left[2^{(n+1)}(B_0 + B_1(n+1) + B_2(n+1)^2)\right]$$
$$+ 2^n(B_0 + B_1 n + B_2 n^2) = 2^n \cdot n^2$$

$$\Rightarrow 2^n [4B_0 + 4nB_1 + 8B_1 + 4B_2 n^2 + 16B_2 n + 16B_2$$
$$- 4B_0 - 4B_1 n - 4B_1 - 4B_2 n^2 - 8B_2 n - 4B_2$$
$$+ B_0 + B_1 n + B_2 n^2] = 2^n \cdot n^2$$

Dividing both sides by 2^n.

$$\Rightarrow 4B_0 + 4nB_1 + 8B_1 + 4B_2 n^2 + 16B_2 n + 16B_2 - 4B_0$$
$$- 4B_1 n - 4B_1 - 4B_2 n^2 - 8B_2 n - 4B_2 + B_0$$
$$+ nB_1 + B_2 n^2 = n^2$$

Equating Coeff. of n^2:
$$4B_2 + B_2 - 4B_2 = 1$$
$$\Rightarrow B_2 = 1$$

Equating Coeff. of n:

$$4B_1 + 16B_2 - 4B_1 - 8B_2 + B_1 = 0$$
$$\Rightarrow 8B_2 + B_1 = 0$$
$$\Rightarrow 8(1) + B_1 = 0$$
$$\Rightarrow B_1 = -8$$

Type (4):

When $\phi(n) = a^n F(n)$

where $F(n)$ is polynomial of degree t.

For P.S trail substitution

$$y_n = a^n (B_0 + B_1 n + B_2 n^2 + \dots + B_t n^t)$$

If 'a' is root of auxiliary equation then above substitution fails.

In this case multiply the trail function by suitable power of n.

If 'a' is single root then multiply by n

If a is double root then multiply by n^2

If a is triple root then multiply by n^3

..... so on.

Example: Solve

$$y_{n+2} - 2y_{n+1} + y_n = 2^n . n^2 \longrightarrow (i)$$

Solution: $y_{n+2} - 2y_{n+1} + y_n = 2^n . n^2$

$$E^2 y_n - 2E y_n + y_n = 2^n . n^2$$

$$(E^2 - 2E + 1) y_n = 2^n . n^2$$

Auxiliary equation is

$$E^2 - 2E + 1 = 0$$

$$(E - 1)^2 = 0$$

$$\Rightarrow E = 1, 1$$

So C.F is $y_n = (C_1 + C_2 n)(1)^n$

$$y_n = C_1 + C_2 n$$

As for P.S put $y_n = 2^n (B_0 + B_1 n + B_2 n^2)$ in (i) we have

$$\Rightarrow 2^{(n+2)} (B_0 + B_1(n+2) + B_2(n+2)^2) - 2\left[2^{(n+1)}(B_0 + B_1(n+1) + B_2(n+1)^2)\right]$$

$$+ 2^n (B_0 + B_1 n + B_2 n^2) = 2^n . n^2$$

$$\Rightarrow 2^n \left[4B_0 + 4nB_1 + 8B_1 + 4B_2 n^2 + 16B_2 n + 16B_2 - 4B_0 - 4B_1 n - 4B_1 - 4B_2 n^2 - 8B_2 n - 4B_2 + B_0 + B_1 n + B_2 n^2\right] = 2^n . n^2$$

Dividing both sides by 2^n

$$\Rightarrow 4B_0 + 4nB_1 + 8B_1 + 4B_2 n^2 + 16B_2 n + 16B_2 - 4B_0 - 4B_1 n - 4B_1 - 4B_2 n^2 - 8B_2 n - 4B_2 + B_0 + nB_1 + B_2 n^2 = n^2$$

Equating Coeff. of n^2:

$$4B_2 + B_2 - 4B_2 = 1$$

$$\Rightarrow B_2 = 1$$

Equating coeff. of n:

$$4B_1 + 16B_2 - 4B_1 - 8B_2 + B_1 = 0$$

$$\Rightarrow 8B_2 + B_1 = 0$$

$$\Rightarrow 8(1) + B_1 = 0$$

$$\Rightarrow B_1 = -8$$

Equating constant term :

$4B_0 + 16B_2 - 4B_2 - 4B_2 + B_0 - 4B_1 = 0$

$\Rightarrow B_0 + 12B_2 + 4B_1 = 0$

$\Rightarrow B_0 + 12(1) + 4(-8) = 0$

$\Rightarrow B_0 + 12 - 32 = 0$

$\Rightarrow B_0 - 20 = 0$

$\Rightarrow B_0 = 20$

P.S is $y_n = 2^n [B_0 + B_1 n + B_2 n^2]$

$= 2^n [20 - 8n + n^2]$

General sol = C.F + P.S

$y_n = c_1 + c_2 n + 2^n (20 - 8n + n^2)$

(Ans)

Example:-

Solve $y_{k+3} - 5y_{k+2} + 8y_{k+1} - 4y_k = 2^k . k$

Solution:-

$y_{k+3} - 5y_{k+2} + 8y_{k+1} - 4y_k = 2^k . k \rightarrow (i)$

$E^3 y_k - 5E^2 y_k + 8E y_k - 4y_k = 2^k . k$

$(E^3 - 5E^2 + 8E - 4) y_k = 2^k . k$

For C.F : Auxiliary equation is

$E^3 - 5E^2 + 8E - 4 = 0$

$(E-1)(E^2 - 4E + 4) = 0$

$\Rightarrow (E-1)(E-2)(E-2) = 0$

$\Rightarrow E = 1, 2, 2$

So C.S is

$y_k = C_1 (1)^k + (C_2 + C_3 k)(2)^k \longrightarrow *$

Put $y_k = k^2 2^k (B_0 + B_1 k)$

($\because k^2$ because 2 is double root)

Put in eq (i)

$(k+3)^2 2^{(k+3)} (B_0 + B_1 (k+3)) - 5 [(k+2)^2 2^{(k+2)} (B_0 + B_1 (k+2))$

$+ 8 [(k+1)^2 2^{k+1} (B_0 + B_1 (k+1)) -$

$4k^2 2^k (B_0 + B_1 k) = 2^k . k$

$2^k [8(k+3)^2 (B_0 + B_1 k + 3B_1) - 5(4)(k+2)^2 (B_0 + B_1 k + 2B_1)$

$+ 8(2)(k+1)^2 (B_0 + B_1 k + B_1) -$

$4k^2 (B_0 + B_1 k)] = 2^k . k$

Dividing by 2^k

$8(k+3)^2 (B_0 + B_1 k + 3B_1) - 20(k+2)^2 (B_0 + B_1 k + 2B_1)$

$+ 16(k+1)^2 (B_0 + B_1 k + B_1) - 4k^2 (B_0 + B_1 k) = k$

Equating Coeff. of like power of k

$48B_0 + 216B_1 - 80B_0 - 240B_1 + 32B_1 + 48B_1 = 1$

$\longrightarrow (2)$

$72B_0 + 216B_1 - 80B_0 - 160B_1 + 16B_1 + 16B_1 = 0$

$\longrightarrow (3)$

$\Rightarrow 24 B_1 = 1$

$$B_1 = \frac{1}{24}$$

③ $\Rightarrow 8B_0 + 72 B_1 = 0$

$$8B_0 + 72\left(\frac{1}{24}\right) = 0$$

$$B_0 = -3/8$$

Particular Solution is

$$y_k = k^2 \cdot 2^k \left(\frac{-3}{8} + \frac{1}{24}k\right)$$

General solution of given difference equation is

$$y_k = C_1(1)^k + (C_2 + C_3 k)2^k + k^2 \cdot 2^k\left(\frac{-3}{8} + \frac{1}{24}k\right)$$

(Ans)

EXERCISE

Solve

(i) $y_{n+2} - 9y_{n+1} + 20y_n = 3^n(n^2+1)$

Ans: $y_n = C_1 4^n + C_2 5^n + \left(\frac{37}{2} + \frac{9}{2}n + \frac{n^2}{2}\right)3^n$

(ii) $y_{k+2} - 2y_{k+1} + y_k = k2^k - 2^k$
$= (k-1)2^k = k2^k - 2^k$

Ans: $y_k = C_1 + C_2 k + (k-5)2^k$

(iii) $U_{n+3} + 8U_n = (2n+3)2^n$

Ans: $U_n = C_1(-2)^n + 2^n\left(C_2 \cos\frac{\pi}{3}n + C_3 \sin\frac{\pi}{3}n\right) + \frac{1}{8}n2^n$

(iv) $y_{k+2} - 13y_{k+1} + 36y_k = 2^k(k^2+1)$

Ans: $C_1 4^k + C_2 9^k + 2^k\left(\frac{123}{343} + \frac{9}{49}k + \frac{1}{14}k^2\right)$

(v) $y_{k+2} + 6y_{k+1} + 25y_k = 2^k + k + 4$

(vi) $(E^2 - 6E + 9)y_k = 3^k \cdot k2$

Ans:

$$y_k = (C_1 + C_2 k)3^k + 3^k k^2\left(\frac{1}{108} - \frac{1}{27}k + \frac{1}{108}k^2\right)$$

Type (5):

when $\phi(n) = a\sin bn$ or $a\cos bn$
also $\phi(n) = \cos bn + \sin bn$

For P.S Trail substitution is
$$y_n = A\sin bn + B\cos bn.$$

If trail substitution contain a term of C.F then multiply the trail substitution by suitable power of n.

Example.
$$y_{n+2} - 4y_{n+1} + 4y_n = \sin n \longrightarrow (i)$$

Solution:
$$y_{n+2} - 4y_{n+1} + 4y_n = \sin n$$
$$E^2 y_n - 4E y_n + 4y_n = \sin n$$
$$(E^2 - 4E + 4)y_n = \sin n$$

Ch. eq for C.S
$$E^2 - 4E + 4 = 0$$
$$(E-2)^2 = 0$$
$$\Rightarrow E = 2, 2$$

C.F is $y_n = (C_1 + C_2 n)(2)^n \longrightarrow (*)$

For P.S Put $y_n = A_1\sin n + A_2\cos n$ in (i) we have

$$A_1\sin(n+2) + A_2\cos(n+2) - 4[A_1\sin(n+1) + A_2\cos(n+1)]$$
$$+ 4[A_1\sin n + A_2\cos n] = \sin n$$

$$\Rightarrow A_1[\sin(n)\cos(2) + \cos(n)\sin(2)] +$$
$$A_2[\cos(n)\cos(2) - \sin(n)\sin(2)] -$$
$$4A_1[\sin(n)\cos(1) + \cos(n)\sin(1)] -$$
$$4A_2[\cos(n)\cos(1) - \sin(n)\sin(1)] +$$
$$4A_1\sin n + 4A_2\cos n = \sin(n)$$

$$\Rightarrow [A_1\cos(2) - A_2\sin(2) - 4A_1\sin(1) - 4A_2\sin(1) + 4A_1]\sin(n)$$
$$+ [A_1\sin(2) + A_2\cos(2) - 4A_1\cos(1) - 4A_2\cos(1) + 4A_2]\cos n = \sin(n)$$

Equating coeff of $\sin(n)$:
$$(\cos 2 - 4\cos 1 + 4)A_1 - (\sin 2 + 4\sin 1)A_2 = 1 \rightarrow (A)$$

Equating Coeff of $\cos(n)$:
$$(\sin 2 - 4\sin 1)A_1 + (\cos 2 - 4\cos 1 + 4)A_2 = 0 \rightarrow (B)$$

Let $K_1 = \cos 2 - 4\cos 1 + 4$
$K_2 = \sin 2 + 4\sin 1$
then (A) and (B) becomes

$$K_1 A_1 - K_2 A_2 = 1 \Rightarrow K_1 A_1 - K_2 A_2 - 1 = 0$$
$$K_2 A_1 - K_1 A_2 = 0 \Rightarrow K_2 A_1 - K_1 A_2 - 0 = 0$$

$$\frac{A_1}{0 + K_1} = \frac{-A_2}{0 + K_2} = \frac{1}{K_1^2 + K_2^2}$$

$$\Rightarrow A_1 = \frac{K_1}{K_1^2 + K_2^2}, \quad A_2 = \frac{-K_2}{K_1^2 + K_2^2}$$

P.S is $y_n = \dfrac{k_1}{k_1^2+k_2^2}\sin n - \dfrac{k_2}{k_1^2+k_2^2}\cos n \rightarrow (**)$

General Solution is (from * and **)

$$y_n = C.F + P.S$$
$$= (C_1 + C_2 n)2^n + \frac{k_1}{k_1^2+k_2^2}\sin n - \frac{k_2}{k_1^2+k_2^2}\cos n$$

where $k_1 = \cos 2 - 4\cos 1 + 4$

$k_2 = \sin 2 - 4\sin 1$

(Ans)

Example:

$$y_{k+2} - 2y_{k+1} + y_k = \sin 5k + \cos 5k + 6$$

Solution: $y_{k+2} - 2y_{k+1} + y_k = \sin 5k + \cos 5k + 6$

$$E^2 y_k - 2Ey_k + y_k = \sin 5k + \cos 5k + 6$$
$$(E^2 - 2E + 1)y_k = \sin 5k + \cos 5k + 6$$

For C.S Auxiliary equation is

$$E^2 - 2E + 1 = 0$$
$$(E-1)^2 = 0$$
$$E = 1, 1$$

$$y_k = (C_1 + C_2 k)(1)^k \longrightarrow (*)$$

For P.S Put $y_k^* = A\sin 5k + B\cos 5k + Ck^2$

Put in $y_{k+2} - 2y_{k+1} + y_k = \sin 5k + \cos 5k + 6$

we have

$A\sin 5(k+2) + B\cos 5(k+2) + C(k+2)^2$
$-2[A\sin 5(k+1) + B\cos 5(k+1) + C(k+1)^2] +$
$A\sin 5k + B\cos 5k + Ck^2 = \sin 5k + \cos 5k + 6$

$\Rightarrow A[\sin 5k\cos 10 + \cos 5k\sin 10] +$
$B[\cos 5k\cos 10 - \sin 5k\sin 10] + C[k^2 + 4k + 4]$
$-2A[\sin 5k\cos 5 + \cos 5k\sin 5] -$
$2B[\cos 5k\cos 5 - \sin 5k\sin 5] - 2C[k^2 + 2k + 1]$
$+ A\sin 5k + B\cos 5k + Ck^2 = \sin 5k + \cos 5k$
$+ 6$

Comparing co-efficients of $\sin 5k$, $\cos 5k$ and constant, we have

$$(\cos 10 - 2\cos 5 + 1)A - (\sin 10 - 2\sin 5)B = 1 \rightarrow (i)$$
$$(\sin 10 - 2\sin 5)A + (\cos 10 - 2\cos 5 + 1)B = 1 \rightarrow (ii)$$

and

$$C(\cancel{k^2} + \cancel{4k} + 4 - \cancel{2k^2} - \cancel{4k} - 2 + \cancel{k^2}) = 6$$
$$C(2) = 6$$
$$\Rightarrow C = 3$$

Now (i) and (ii) written as

$$k_1 A - k_2 B = 1 \Rightarrow k_1 A - k_2 B - 1 = 0$$
$$k_2 A + k_1 B = 1 \Rightarrow k_2 A + k_1 B - 1 = 0$$

Where $k_1 = \cos 10 - 2\cos 5 + 1$

$k_2 = \sin 10 - 2\sin 5$

Solving these eq. by cross- multiplication

$$\frac{A}{K_1+K_2} = \frac{-B}{-K_2+K_1} = \frac{1}{K_1^2+K_2^2}$$

$$\Rightarrow A = \frac{K_1+K_2}{K_1^2+K_2^2}, \quad B = \frac{K_1-K_2}{K_1^2+K_2^2}$$

So P.S is

$$y_k = \left(\frac{K_1+K_2}{K_1^2+K_2^2}\right)\sin 5k + \left(\frac{K_1-K_2}{K_1^2+K_2^2}\right)\cos 5k + 3k^2 \longrightarrow (*')$$

The general solution by $(*)$ and $(*')$

$$y_k = (C_1+C_2k)(1)^k + \left(\frac{K_1+K_2}{K_1^2+K_2^2}\right)\sin 5k + \left(\frac{K_1-K_2}{K_1^2+K_2^2}\right)\cos 5k + 3k^2$$

Where $K_1 = \cos 10 - 2\cos 5 + 1$

$K_2 = \sin 10 - 2\sin 5$

(Ans)

EXERCISE

Solve the following difference Equation

1) $y_{n+2} - 3y_{n+1} - 4y_n = \sin 2n$

Ans: $y_n = C_1 4^n + C_2(-1)^n + \frac{K_1}{K_1^2+K_2^2}\sin 2n - \frac{K_2}{K_1^2+K_2^2}\cos 2n$

where $K_1 = \cos 4 - 3\cos 2 - 4$

$K_2 = \sin 4 - 3\sin 2$

2) $y_{n+2} - 2y_{n+1} + y_n = \sin 5n + \cos 5n + 9$

3) $y_{n+2} + y_n = \sin \frac{n\pi}{2}$

Ans: $C_1 \cos \frac{n\pi}{2} + C_2 \sin \frac{n\pi}{2} - \frac{1}{2} n \sin \frac{n\pi}{2}$

4) $y_{n+2} - 8y_{n+1} + y_n = 2^n + \sin n$

5) $y_{n+4} - 16 y_n = \sin 3n$

6) $y_{n+2} + y_n = \sin x$

7) $y_{n+2} + y_n = \cos \frac{n}{2}$

8) $y_{k+2} - 7y_{k+1} + 12y_k = \cos k$

Type (6):

when $\phi(n) = a^n (\cos An \text{ or } \sin An)$. where A is constant.

In order to find particular solution we shall make substitution

$$y_n = a^n (C_1 \sin An + C_2 \cos An)$$

and find value of C_1 and C_2

Example:

$$y_{t+2} - 7y_{t+1} - 8y_t = 7^t(\cos 3t + \sin 3t) \rightarrow (i)$$

Solution:- $y_{t+2} - 7y_{t+1} - 8y_t = 7^t(\cos 3t + \sin 3t)$

$\Rightarrow E^2 y_t - 7E y_t - 8y_t = 7^t(\cos 3t + \sin 3t)$

$\Rightarrow (E^2 - 7E - 8) y_t = 7^t(\cos 3t + \sin 3t)$

Auxiliary equation is

$E^2 - 7E - 8 = 0$

$\Rightarrow E^2 - 8E + E - 8 = 0$

$\Rightarrow E(E-8) + 1(E-8) = 0$

$\Rightarrow (E-8)(E+1) = 0$

$\Rightarrow E = -1, 8$

C.F is $y_t = C_1(-1)^t + C_2(8)^t \rightarrow (*)$

For P.S

put $y_t = 7^t [A_1 \cos 3t + A_2 \sin 3t]$

put in eq (i)

$7^{(t+2)} [A_1 \cos 3(t+2) + A_2 \sin 3(t+2)]$

$\Rightarrow - 7 \cdot 7^{t+1} [A_1 \cos 3(t+1) + A_2 \sin 3(t+1)]$

$-8 [7^t (A_1 \cos 3t + A_2 \sin 3t)] = 7^t(\cos 3t + \sin 3t)$

$\Rightarrow$ Dividing both sides by 7^t

$\Rightarrow 49 A_1 [\cos 3t \cos 6 - \sin 3t \sin 6] +$
$49 A_2 [\sin 3t \cos 6 + \cos 3t \sin 6] -$
$49 A_1 [\cos 3t \cos 3 - \sin 3t \sin 3] -$
$49 A_2 [\sin 3t \cos 3 - \cos 3t \sin 3]$
$- 8 A_1 \cos 3t - 8 A_2 \sin 3t = \cos 3t + \sin 3t$

$\Rightarrow$ Comparing Coeffs of $\cos 3t$ and $\sin 3t$

$(49 \cos 6 - 49 \cos 3 - 8) A_1 + (49 \sin 6 - 49 \sin 3) A_2 = 1$

$-(49 \sin 6 - 49 \sin 3) A_1 + (49 \cos 6 - 49 \cos 3 - 8) A_2 = 1$

Let $K_1 = 49 \cos 6 - 49 \cos 3 - 8$

$K_2 = 49 \sin 6 - 49 \sin \theta$

$\Rightarrow K_1 A_1 + K_2 A_2 - 1 = 0$

$-K_2 A_1 + K_1 A_2 - 1 = 0$

$$\frac{A_1}{-K_2 + K_1} = \frac{-A_2}{-K_1 - K_2} = \frac{1}{K_1^2 + K_2^2}$$

$$\Rightarrow A_1 = \frac{K_1 - K_2}{K_1^2 + K_2^2}, \quad A_2 = \frac{K_1 + K_2}{K_1^2 + K_2^2}$$

So P.S is

$$y_t = 7^t\left[\left(\frac{k_1-k_2}{k_1^2+k_2^2}\right)\cos 3t + \left(\frac{k_1+k_2}{k_1^2+k_2^2}\right)\sin 3t\right] \to (*')$$

By $*$ and $*'$ the general solution is

$$y_t = C.F + P.S$$

$$= C_2 8^t + C_1(-1)^t + 7^t\left[\frac{(k_1-k_2)\cos 3t + (k_1+k_2)\sin 3t}{k_1^2+k_2^2}\right]$$

(Ans)

Example:-

Solve $y_{k+2} + 13y_{k+1} + 3y_k = 3^k\cos 4k \to (i)$

Solution:-

Given $y_{k+2} + 13y_{k+1} + 3y_k = 3^k\cos 4k$

$$E^2 y_k + 13Ey_k + 3y_k = 3^k\cos 4k$$

$$(E^2 + 13E + 3)y_k = 3^k\cos 4k.$$

Auxiliary equation

$$E^2 + 13E + 3 = 0$$

$$E = \frac{-13 \pm \sqrt{9 - 4(13)(1)}}{2}$$

$$= \frac{-13 \pm \sqrt{157}}{2}$$

So complementry Solution is

$$y_k = A_1\left(\frac{-13+\sqrt{157}}{2}\right)^k + A_2\left(\frac{-13-\sqrt{157}}{2}\right)^k \to (*)$$

For P.S put $y_k = 3^k[C_1\sin 4k + C_2\cos 4k]$

put in (i) we have

$$\Rightarrow 3^{k+2}[C_1\sin 4(k+2) + C_2\cos 4(k+2)] + 13\cdot 3^{k+1}[C_1\sin 4(k+1) + C_2\cos 4(k+1)] + 3\cdot 3^k[C_1\sin 4k + C_2\cos 4k] = 3^k\cos 4k$$

$\Rightarrow$ Dividing both sides by 3^k we get

$$\Rightarrow (3)^2 C_1[\sin 4k\cos 8 + \cos 4k\sin 8] +$$
$$9C_2[\cos 4k\cos 8 - \sin 4k\sin 8] +$$
$$39C_1[\sin 4k\cos 4 + \cos 4k\sin 4] +$$
$$39C_2[\cos 4k\cos 4 - \sin 4k\sin 4] +$$
$$3C_1\sin 4k + 3C_2\cos 4k = \cos 4k$$

Comparing co-efficient of $\cos 4k$ and $\sin 4k$.

$$C_1[9\cos 8 + 39\cos 4 + 3] - C_2[9\sin 8 + 39\sin 4] = 0$$
$$C_1[9\sin 8 + 39\sin 4] + C_2[9\cos 8 + 39\cos 4 + 3] = 1$$

Let $9\cos 8 + 39\cos 4 + 3 = k_1$

$9\sin 8 + 39\sin 4 = k_2$

then

$$k_1C_1 - k_2C_2 - 0 = 0$$
$$k_2C_1 + k_1C_2 - 1 = 0.$$

$$\frac{C_1}{k_2} = \frac{-C_2}{-k_1} = \frac{1}{k_1^2+k_2^2}$$

So $C_1 = \dfrac{k_2}{k_1^2+k_2^2}$, $C_2 = \dfrac{k_1}{k_1^2+k_2^2}$

So P.S is

$$y_k = 3^k \left[\frac{k_2 \sin 4k + k_1 \cos 4k}{k_1^2 + k_2^2} \right] \rightarrow (*')$$

by * and *' general solution is

$$y_k = A_1 \left(\frac{-13 + \sqrt{157}}{2} \right) + A_2 \left(\frac{-13 - \sqrt{157}}{2} \right) + 3^k \left[\frac{k_2 \sin 4k + k_1 \cos 4k}{k_1^2 + k_2^2} \right]$$

(Ans)

Example:

$$y_{k+2} + y_{k+1} + y_k = 2^k . \sin k \cos 3k$$

Solution:

Given $y_{k+2} + y_{k+1} + y_k = 2^k . \sin k \cos 3t$

$$y_{k+2} + y_{k+1} + y_k = \frac{2^k}{2} [2 \sin k \cos 3t]$$

$$y_{k+2} + y_{k+1} + y_k = \frac{2^k}{2} [\sin 2k - \sin k] \longrightarrow (1)$$

For C.F

$$y_{k+2} + y_{k+1} + y_k = 0$$

$$E^2 y_k + E y_k + y_k = 0$$

$$(E^2 + E + 1) y_k = 0 \quad \text{As } y_k \neq 0$$

$$\Rightarrow E^2 + E + 1 = 0$$

$$E = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3} i}{2}$$

∴ complementry sol is

$$y_k = R^k [A_1 \cos k\theta + A_2 \sin k\theta]$$

where $R = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$

$$\theta = \tan^{-1} \frac{\sqrt{3}}{-1} = 120^\circ$$

$$y_k = (1)^k [A_1 \cos 120 k + A_2 \sin 120 k] \longrightarrow (*)$$

For particular solution we divide (1) into two equations.

$$y_{k+2} + y_{k+1} + y_k = 2^k . \frac{1}{2} \sin 2k \longrightarrow (A)$$

and $y_{k+2} + y_{k+1} + y_k = 2^k . \frac{1}{2} \sin k . \longrightarrow (B)$

First we solve (A)

put $y_k = 2^k (c_1 \sin 2k + c_2 \cos 2k)$

(A) ⇒

$$2^{k+2} (c_1 \sin 2(k+2) + c_2 \cos(k+2)) + 2^{k+1} (c_1 \sin 2(k+1) + c_2 \cos(k+1)) + 2^k (c_1 \sin 2k + c_2 \cos 2k) = 2^k . \frac{1}{2} \sin 2k$$

Dividing by 2^k and comparing coeff. of $\sin 2k$ and $\cos 2k$ we have

$$(4 \cos 4 + 2 \cos 2 + 1) c_1 - (4 \sin 4 + 2 \sin 2) c_2 = \tfrac{1}{2}$$

$$(4 \sin 4 + 2 \sin 2) c_1 + (4 \cos 4 + 2 \cos 2 + 1) c_2 = 0$$

So P.S is

$$y_k = 3^k\left[\frac{k_2\sin 4k + k_1\cos 4k}{k_1^2 + k_2^2}\right] \rightarrow (*')$$

by * and *' general solution is

$$y_k = A_1\left(\frac{-13+\sqrt{157}}{2}\right) + A_2\left(\frac{-13-\sqrt{157}}{2}\right) + 3^k\left[\frac{k_2\sin 4k + k_1\cos 4k}{k_1^2 + k_2^2}\right]$$

(Ans).

Example:

$$y_{k+2} + y_{k+1} + y_k = 2^k.\sin k\cos 3k.$$

Solution:

Given $y_{k+2} + y_{k+1} + y_k = 2^k.\sin k\cos 3t$

$$y_{k+2} + y_{k+1} + y_k = \frac{2^k}{2}[2\sin k\cos 3t]$$

$$y_{k+2} + y_{k+1} + y_k = \frac{2^k}{2}[\sin 2k - \sin k] \longrightarrow ①$$

For C.F

$$y_{k+2} + y_{k+1} + y_k = 0$$

$$E^2 y_k + E y_k + y_k = 0$$

$$(E^2 + E + 1) y_k = 0 \quad \text{As } y_k \neq 0$$

$$\Rightarrow E^2 + E + 1 = 0$$

$$E = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}\,i}{2}$$

So complementry sol is

$$y_k = R^k[A_1\cos k\theta + A_2\sin k\theta]$$

where $R = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$

$$\theta = \tan^{-1}\frac{\sqrt{3}}{-1} = 120^\circ$$

$$y_k = (1)^k[A_1\cos 120k + A_2\sin 120k] \rightarrow (*).$$

For particular solution we divide (1) into two equations.

$$y_{k+2} + y_{k+1} + y_k = 2^k.\frac{1}{2}\sin 2k \rightarrow Ⓐ$$

and $y_{k+2} + y_{k+1} + y_k = 2^k.\frac{1}{2}\sin k. \rightarrow$ Ⓑ

First we solve (A)

put $y_k = 2^k(c_1\sin 2k + c_2\cos 2k)$

(A) =>

$$2^{k+2}(c_1\sin 2(k+2) + c_2\cos(k+2)) + 2^{k+1}(c_1\sin 2(k+1) + c_2\cos(k+1)) + 2^k(c_1\sin 2k + c_2\cos 2k) = 2^k.\frac{1}{2}\sin 2k.$$

Dividing by 2^k and comparing coeff. of $\sin 2k$ and $\cos 2k$ we have

$$(4\cos 4 + 2\cos 2 + 1)c_1 - (4\sin 4 + 2\sin 2)c_2 = 1/2$$

$$(4\sin 4 + 2\sin 2)c_1 + (4\cos 4 + 2\cos 2 + 1)c_2 = 0$$

Let $k_1 = 4\cos 4 + 2\cos 2 - 1$
$k_2 = 4\sin 4 + 2\sin 2$
then above write as

$$k_1 C_1 - k_2 C_2 = 1/2$$
$$k_2 C_1 + k_1 C_2 = 0$$

$$\Rightarrow k_1 C_1 - k_2 C_2 - 1/2 = 0$$
$$k_2 C_1 + k_1 C_2 - 0 = 0$$

by solving

$$\frac{C_1}{-k_1/2} = \frac{-C_2}{-k_2/2} = \frac{1}{k_1^2 + k_2^2}$$

$$\Rightarrow C_1 = \frac{k_1}{2(k_1^2+k_2^2)}, \quad C_2 = \frac{-k_2}{2(k_1^2+k_2^2)}$$

So particular solution is

$$y_k = 2^k \left[\frac{k_1 \sin 2k - k_2 \cos 2k}{2(k_1^2+k_2^2)} \right]$$

Similarly For (B) P.S is

$$y_k = 2^k \left[\frac{k_1 \sin k - k_2 \cos k}{2(k_1^2+k_2^2)} \right]$$

Particular solution of (A) and (B) is actually P.S of (i)

So P.S of (i) = P.S of (A) − P.S of (B)

$$y_k = 2^k \left[\frac{k_1(\sin 2k - \sin k) - k_2(\cos 2k - \cos k)}{2(k_1^2+k_2^2)} \right] \rightarrow (*')$$

General solution of eq (i) is (by * and *')

$$y_k = (1)^k \left[A_1 \cos 120^\circ k + A_2 \sin 120 k \right] + 2^k \left[\frac{k_1(\sin 2k - \sin k) - k_2(\cos 2k - \cos k)}{2(k_1^2+k_2^2)} \right]$$

(Answer).

N

Simultaneous Linear Difference Equations

If two or more difference equations are given with same number of unknown functions, then we can solve such equations simultaneously by using a procedure which eliminates all but one of the unknown.

Example: Solve the system

$$U_{n+1} - U_n + 3V_n = 7$$
$$3V_{n+1} + V_n - 2U_n = 6$$

Solution:

Given $U_{n+1} - U_n + 3V_n = 7$

$$3V_{n+1} - V_n - 2U_n = 6$$

In operator notation, the above equations can be written as

$$EU_n - U_n + 3V_n = 7$$
$$\Rightarrow (E-1)U_n + 3V_n = 7 \longrightarrow (i)$$

and $3V_{n+1} + V_n - 2U_n = 6$

$$3EV_n + V_n - 2U_n = 6$$
$$(3E+1)V_n - 2U_n = 6 \longrightarrow (ii)$$

Multiplying eq (i) by $(3E+1)$ and eq (ii) by 3 then subtract them we get

$$(3E+1)(E-1)U_n + \cancel{3(3E+1)V_n} = 7(3E+1)$$
$$\underset{+}{-}\,6U_n \underset{-}{+} \cancel{3(3E+1)V_n} = \underset{-}{+}\,18$$

$$(3E^2-2E-1)U_n + 6U_n = 21E + 7 - 18$$
$$(3E^2-2E+5)U_n = 21 + 7 - 18$$
$$(3E^2-2E+5)U_n = 10$$
$$\Rightarrow 3U_{n+2} - 2U_{n+1} + 5U_n = 10 \longrightarrow (3)$$

which is a difference equation (Non-homogenuous) to solve this we find C.S and P.S of (3)

For complemetry Solution

$$(3E^2-2E+5)U_n = 0$$
$$\Rightarrow 3E^2 - 2E + 5 = 0$$
$$\Rightarrow E = \frac{2 \pm \sqrt{4-4(3)(5)}}{6}$$
$$E = \frac{2 \pm \sqrt{56}}{6} = \frac{1 \pm \sqrt{14}\,i}{3}$$

So C.S is

$$U_n = R^n\left[A_1\cos n\theta + A_2\sin n\theta\right]$$

$$R = \sqrt{\frac{1}{9} + \frac{14}{9}} = \sqrt{\frac{15}{9}} = \frac{\sqrt{15}}{3}$$

$$\theta = \tan^{-1}(\sqrt{14})$$

$$\Rightarrow U_n = \left(\frac{\sqrt{15}}{3}\right)^n\left[A_1\cos n\theta + A_2\sin n\theta\right]$$
$$\because \theta = \tan^{-1}(\sqrt{14})$$

To find particular solution of (3) Put

$U_n^* = C$

$(3) \Rightarrow 3C - 2C + 5C = 10$

$6C = 10$

$C = 5/3$

$U_n^* = 5/3$

General Solution $= U_n + U_n^*$

$= \left(\frac{\sqrt{15}}{3}\right)^n \left[A_1 \cos n\theta + A_2 \sin n\theta\right] + 5/3$

where $\theta = \tan^{-1}(\sqrt{14})$.

Which is solution of U_n. (Ans).

To find V_n, Put value of U_n in (i)

$3V_n = 7 - (E-1)U_n$

$= 7 - U_{n+1} + U_n$

$V_n = \frac{7}{3} - \frac{1}{3}\left(\frac{\sqrt{15}}{3}\right)^{n+1}\left[A_1\cos(n+1)\theta + A_2 \sin(n+1)\theta\right] + 5/9$

$+ \left(\frac{\sqrt{15}}{3}\right)^n \left[A_1\cos(n)\theta + A_2 \sin n\theta\right] + \frac{5}{3}$

Which is required solution for V_n.

(Ans)

Example:

Solve the system of Equation

$U_{n+1} + V_n - 3U_n = n$

$3U_n + V_{n+1} - 5V_n = 4^n$

Solution:- Given

$U_{n+1} + V_n - 3U_n = n \rightarrow (i)$

$V_{n+1} - 5V_n + 3U_n = 4^n \rightarrow (ii)$

In operator notation, above eq (i) & (ii) becomes

$EU_n + V_n - 3U_n = n$

$EV_n - 5V_n + 3U_n = 4^n$

and $(E-3)U_n + V_n = n \rightarrow (iii)$

$(E-5)V_n + 3U_n = 4^n \rightarrow (iv)$

Multiplying (iii) by $(E-5)$ then subtract from (iv)

$(E-5)(E-3)U_n + \cancel{(E-5)V_n} = (E-5)n$

$\underset{-}{+3}\ \ U_n \underset{-}{+} \cancel{(E-5)V_n} = \underset{-}{+}4^n$

$(E^2 - 8E + 15)U_n - 3U_n = -5n + En - 4^n$

$(E^2 - 8E + 12)U_n = n + 1 - 5n - 4^n$

$(\because En = n+1)$

$(E^2 - 8E + 12)U_n = 1 - 4n - 4^n \rightarrow (V)$

Auxiliary Equation is $E^2 - 8E + 12 = 0$

$E^2 - 2E - 6E + 12 = 0$

$E(E-2) - 6(E-2) = 0$

$(E-2)(E-6) = 0$

$\Rightarrow E = 2, 6$

So C.F is

$$U_n = C_1 (2)^n + C_2 (6)^n \longrightarrow (*)$$

Eq (V) be write it as

$U_{n+2} - 8U_{n+1} + 12U_n = 1 - 4n - 4^n$

For P.S consider two equations

$U_{n+2} - 8U_{n+1} + 12U_n = 1 - 4n \longrightarrow (A)$

$U_{n+2} - 8U_{n+1} + 12U_n = -4^n \longrightarrow (B)$

Now consider (A) For P.S

Put $U_n = B_0 + B_1 n$

Put in (A)

$\Rightarrow B_0 + B_1(n+2) - 8B_0 - 8B_1(n+1) + 12B_0 + 12B_1 n = 1 - 4n$

$\Rightarrow B_0 + B_1 n + 2B_1 - 8B_0 - 8B_1 n - 8B_1 + 12B_0 + 12B_1 n = 1 - 4n$

Equating coeff of n & constant term

$B_1 - 8B_1 + 12B_1 = -4$

$5B_1 = -4$

$B_1 = -4/5$

and

$B_0 + 2B_1 - 8B_0 - 8B_1 + 12B_0 = 1$

$5B_0 - 6B_1 = 1$

$5B_0 - 6\left(\frac{-4}{5}\right) = 1$

$5B_0 = 1 - \frac{24}{5}$

$B_0 = \frac{-19}{25}$

So P.S for (A) is

$$U_n = \frac{-19}{25} - \frac{4}{5} n$$

For eq (B) Let $U_n = C 4^n$

$U_{n+1} = C 4^{n+1}$

$U_{n+2} = C 4^{n+2}$

put in (B)

$C4^{n+2} - 8C4^{n+1} + 12C4^n = -4^n$

$4^n(16C - 32C + 12C) = 4^n(-1)$

$16C - 32C + 12C = -1$

$-4C = -1$

$C = 1/4$

P.S is $U_n = \frac{1}{4} 4^n$

So P.S for (V) is = P.S of (A) + P.S of (B)

$$U_n = \frac{-19}{25} - \frac{4}{5} n + \frac{1}{4} 4^n \longrightarrow (*')$$

General solution of U_n is (by $*$ & $*'$)

$$U_n = C_1 (2)^n + C_2 (6)^n - \frac{19}{25} - \frac{4}{5} n + \frac{1}{4} 4^n$$

which is required solution for U_n.

Now; by eq (iii)

$$V_n = n - U_{n+1} - 3U_n$$

$$= n - \left[C_1 (2)^{n+1} + C_2 (6)^{n+1} - \frac{19}{25} - \frac{4}{5}(n+1) + \frac{1}{4} 4^{(n+1)}\right]$$

$$- 3\left[C_1 (2)^n + C_2 (6)^n - \frac{19}{25} - \frac{4}{5} n + \frac{1}{4} 4^n\right]$$

$$V_n = n + (-4C_1 + 3C_1)2^n + (-6C_2 + 3C_2)5^n + \left(\frac{19}{125} - \frac{27}{25} + \frac{5}{5}\right) + \left(\frac{2}{5} - \frac{13}{5}\right)n + \left(-1 + \frac{3}{4}\right)4^n$$

$$V_n = n + C_1 2^n - 3C_2 5^n - \frac{18}{25} - \frac{8}{5}n - \frac{4^n}{4}$$

$$V_n = C_1 2^n - 3C_2 5^n - \frac{3}{5}n - \frac{18}{25} - \frac{4^n}{4}$$

which is required solution for V_n.

(Answer)

Example: Solve system of equations

$$U_{n+1} - U_n + V_n = 7$$
$$3V_{n+1} - 2V_n + U_n = 2$$

Solution:- Given $U_{n+1} - U_n + V_n = 7$

$$3V_{n+1} - 2V_n + U_n = 2$$

In operator notation, we can write

$$EU_n - U_n + V_n = 7$$
$$\Rightarrow (E-1)U_n + V_n = 7 \rightarrow (i)$$
$$\Rightarrow (3E-2)V_n + U_n = 2 \rightarrow (ii)$$

Multiplying (ii) by $(E-1)$ and subtract from (i)

$$(3E-2)(E-1)V_n + \cancel{(E-1)U_n} = 2(E-1)$$
$$\underset{-}{+}V_n \underset{-}{+} \cancel{(E-1)U_n} = \underset{-}{+}7$$

$$(3E^2 - 5E + 2)V_n - V_n = 2 - 2 - 7$$

$$\Rightarrow (3E^2 - 5E + 1)V_n = -7$$
$$3V_{n+2} - 5V_{n+1} + V_n = -7 \rightarrow (iii)$$

Auxiliary equation is $3E^2 - 5E + 1 = 0$

$$\Rightarrow E = \frac{5 \pm \sqrt{13}}{6}$$

then C.F $V_n = C_1\left(\frac{5+\sqrt{13}}{6}\right)^n + C_2\left(\frac{5-\sqrt{13}}{6}\right)^n$

To find P.S Put $V_n = C$ in (iii)

$$3C - 5C + C = -7$$
$$-C = -7$$
$$C = 7$$

P.S of $V_n = 7$

General Solution = C.F + P.S

$$= C_1\left(\frac{5+\sqrt{13}}{6}\right)^n + C_2\left(\frac{5-\sqrt{13}}{6}\right)^n + 7$$

which is required solution for V_n.

Now

$$U_n = 2 - 3V_{n+1} + 2V_n$$
$$= 2 - 3\left[C_1\left(\frac{5+\sqrt{13}}{6}\right)^{n+1} + C_2\left(\frac{5-\sqrt{13}}{6}\right)^{n+1} + 7\right]$$
$$+ 2\left[C_1\left(\frac{5+\sqrt{13}}{6}\right)^n + C_2\left(\frac{5-\sqrt{13}}{6}\right)^n + 7\right]$$

$$U_n = 2 - 3\left[C_1\left(\frac{5+\sqrt{13}}{6}\right)^{n+1} + C_2\left(\frac{5-\sqrt{13}}{6}\right)^{n+1}\right]$$
$$+ 2\left[C_1\left(\frac{5+\sqrt{13}}{6}\right)^n + C_2\left(\frac{5-\sqrt{13}}{6}\right)^n\right] - 7$$

which is required solution for U_n.

(Answer)

EXERCISE

Solve the following Simulteneous differance equation

(i) $3U_n + 4V_{n+1} - 5V_n = 2^n$
$2U_{n+1} + 3U_n + 4V_n = 7$

(ii) $U_{n+1} - U_n + V_n = 2^n$
$3V_{n+1} + 2V_n + U_n = 7$

(iii) $3V_{n+1} - 2V_n + U_n = n^2$
$U_{n+1} - 3U_n + V_n = 3^n$

(iv) $U_{n+1} - U_n + V_n = 1$
$3V_{n+1} - 2V_n + U_n = 2$

NUMERICAL METHODS

FOR ORDINARY DIFFERENTIAL EQUATION

An ordinary differential equation is a relation between a function, its derivatives and variable upon which they depend.

The most general form of an O.D.E is $\phi(y, y', y'', \ldots\ldots, y^{(n)}) = 0$. where y and its derivatives are all function of x.

In this chapter we shall derive and analyse numarical methods for solving problem of ordinary diff. equation.

The main form of problem that we shall study is initial value problem,

$$y' = f(x, y) \quad ; \quad y(x_0) = y_0$$

The important methods of solving O.D.E's of first order are as follows.

(i) Euler's Method.
(ii) Heun's Method (Improved Euler's MHD)
(iii) Modified Euler's Method.
(iv) Taylor's Series Method.
(v) Runge - Kutta Method.
(vi) Adams - Bashforth method.

etc.

1. EULER'S METHOD :-

Consider the first order differential equation together with an initial conditions is

$$\frac{dy}{dx} = f(x, y) \longrightarrow (i) \quad ; \quad y(x_0) = y_0$$

By Taylor Series

$$y(x_0 + h) = y(x_0) + h y'(x_0) + \frac{h^2}{2!} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \dots$$

$$y(x_0 + h) = y(x_0) + h y'(x_0)$$

∵ Neglecting h^2 and higher power of h.

$$y(x_1) = y(x_0) + h f(x_0, y_0)$$

$$y_1 = y_0 + h f(x_0, y_0)$$

Similarly,

$$y_2 = y_1 + h f(x_1, y_1)$$

$$y_3 = y_2 + h f(x_2, y_2)$$

$$\vdots$$

$$y_{n+1} = y_n + h f(x_n, y_n)$$

It is Euler's Iterative Formula

Remark:

Drawback of Euler's Method, if h is small (h is unit step size), Euler's Method is too slow. However is h is large this method is inaccurate

Example :-

Solve $\frac{dy}{dx} = x + y$ where $y(0) = 1$, $h = 0.2$

Find $y(0.4)$ by Euler's Method and Compare with exact value.

Solution :-

The given initial value problem is

$$\frac{dy}{dx} = x + y \longrightarrow (i) \quad ; \quad y(0) = 1, \; h = 0.2$$

The Euler's Formula is

$$y_{n+1} = y_n + h f(x_n, y_n)$$

As $f(x, y) = x + y$

$$f(x_n, y_n) = x_n + y_n$$

So

$$y_{n+1} = y_n + h(x_n + y_n) \longrightarrow (ii)$$

For $n = 0$, we have From (ii)

$$y_1 = y_0 + h(x_0 + y_0) \qquad , \qquad y_1 = y(x_1)$$

$$= 1 + 0.2(0 + 1) \qquad x_1 = x_0 + h$$

$$= 1 + 0.2 \qquad = 0 + 0.2$$

$$= 1.2 \qquad x_1 = 0.2$$

$$y(0.2) = 1.2$$

For $n = 1$:

$$y_2 = y_1 + h(x_1 + y_1)$$

$$= 1.2 + 0.2(0.2 + 1.2)$$

$$= 1.2 + 0.2(1.4) \qquad x_2 = x_0 + 2h$$

$$= 1.2 + 0.28 \qquad x_2 = 0 + 2(0.2)$$

$$= 1.48 \qquad x_2 = 0.4$$

$$y(0.4) = 1.48$$

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Exact Solution:

Give O.D.E is

$$\frac{dy}{dx} = x + y \rightarrow ③$$

$$\frac{dy}{dx} - y = x \rightarrow ③' \text{ (Linear Diff. EQ)}$$

I.F is $e^{\int -1\,dx} = e^{-x}$

Then ③' becomes

$$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}x$$

$$\frac{d}{dx}\left(ye^{-x}\right) = xe^{-x}$$

Integrating both sides

$$ye^{-x} = -xe^{-x} - e^{-x} + c$$

$$\Rightarrow ye^{-x} = -e^{-x}(x+1) + c$$

$$\Rightarrow y = -(x+1) + ce^{x}$$

$$\Rightarrow y = ce^{x} - (x+1) \rightarrow ④$$

Apply condition $y(0) = 1$

$$1 = c - 1$$

$$\Rightarrow c = 2$$

then ④ becomes

$$y = 2e^{x} - (x+1)$$

at $x = 0.4$

$$y = 2e^{(0.4)} - (0.4+1)$$

$$y = 1.5836.$$

So error = Exact - Apprx. value

$$= 1.5836 - 1.418$$

$$E = 0.1036 \quad \text{(Answer)}$$

Example:

Find solution of $\frac{dy}{dx} = +2xy^2$.

$y(0) = 1$ & $h = 0.05$

Find $y(0.3)$ by using Euler's Method.

Solution:- The given initial value problem is

$$\frac{dy}{dx} = +2xy^2 \longrightarrow ①$$

$y(0) = 1$ i-e $x_0 = 0$, $y_0 = 1$

$$h = 0.05$$

Euler's Formula is

$$y_{n+1} = y_n + hf(x_n, y_n)$$

where $f(x, y) = +2xy^2$

$$f(x_n, y_n) = 2x_n y_n^2$$

then

$$y_{n+1} = y_n + h(2x_n y_n^2) \rightarrow ②$$

n	x_n
0	0
1	0.05
2	0.1
3	0.1
4	0.
5	0.
6	0.

Put $n=0$ in ②:

$$y_1 = y_0 + h(2x_0 y_0^2)$$

$$= 1 + 0.05(2(0)(1)^2)$$

$$= 1 + 0.05(0)$$

$$y(x_1) = 1$$

$y(x_1) = y_1$

$x_1 = x_0 + h$

$= 0 + 0.05 = 0.0$

$$\Rightarrow y(0.05) = 1$$

Put $n = 1$ in ②:

$$y_2 = y_1 + h(2x_1 y_1^2)$$

$$= 1 + 0.05(2(0.05)(1)^2)$$

$$= 1 + 0.05$$

$$y_2 = 1.0050$$

and $x_2 = x_0 + 2h$

$$= 0 + 2(0.05) = 0.100$$

$\Rightarrow y(x_2) = y_2$
$\Rightarrow y(0.100) = 1.0050$

Put $n=2$ in ②:

$$y_3 = y_2 + h(2x_2 y_2^2)$$
$$= 1.0050 + 0.05(2(0.100)(1.0050)^2)$$
$$= 1.0151$$

and $x_3 = x_0 + 3h$
$= 0 + 3(0.05)$
$x_3 = 0.1500$
$\Rightarrow y(0.1500) = 1.0151$

Put $n=3$ in ②:

$$y_4 = y_3 + h(2x_3 y_3^2)$$
$$= 1.0151 + 0.05(2(0.1500)(1.0151)^2)$$
$$= 1.0306$$

and $x_4 = x_0 + 4h$
$= 0 + 4(0.05)$
$x_4 = 0.20$
$y(x_4) = y_4$
$y(0.20) = 1.0306$

Put $n=4$ in ②:

$$y_5 = y_4 + h(2x_4 y_4^2)$$
$$= 1.0306 + 0.05(2(0.20)(1.0306)^2)$$
$$= 1.0518$$

and

$x_5 = x_0 + 5h$
$= 0 + 5(0.05) = 0.2500$
$y(0.25) = 1.0518$

Put $n=5$ in ②:

$$y_6 = y_5 + h(2x_5 y_5^2)$$
$$= 1.0518 + 0.05(2(0.25)(1.0518)^2)$$
$$= 1.0795$$

and

$x_6 = x_0 + 6h$
$= 0 + 6(0.05) = 0.30$
$\Rightarrow y(0.3) = 1.0795$

(Answer)

Example :- Find an approximate value for the solution of initial value problem
$$y' = x + y^2, \quad y(1) = 0.$$
at $x = 1, 1.1, 1.2, 1.3, 1.4, 1.5$.

Solution :- The initial value problem is
$$y' = x + y^2 \quad ; \quad y(1) = 0$$
The Euler's Formula is

$$y_{n+1} = y_n + h f(x_n, y_n) \rightarrow (i)$$

As $f(x,y) = x + y^2$
$f(x_n, y_n) = x_n + y_n^2$

$\Rightarrow$ (i) becomes

$$y_{n+1} = y_n + h(x_n + y_n^2) \longrightarrow ②.$$

In tabular form we can write as

n	x_n	y_n	$y_{n+1} = y_n + h(x_n + y_n^2)$
0	1	0	$y_1 = 0 + 0.1(1 + 0^2) = 0.10$
1	1.1	0.10	$y_2 = 0.10 + 0.1(1.1 + (0.10)^2)$ $= 0.2110$
2	1.2	0.2110	$y_3 = 0.2100 + 0.1(1.2 + (0.211)^2)$ $= 0.33$
3	1.3	0.33	$y_4 = 0.33 + 0.1(1.3 + (0.33)^2)$ $= 0.4709$
4	1.4	0.47	$y_5 = 0.47 + 0.1(1.4 + (0.47)^2)$ $= 0.6321$
5	1.5		

$\Rightarrow y(1.5) = 0.6321$

(Answer)

II HEUN'S METHOD (OR) Improved Euler's Method.

Given $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$

where h is step-size. Integrate from x_n to x_{n+1}

$$\int_{x_n}^{x_{n+1}} \frac{dy}{dx} = \int_{x_n}^{x_{n+1}} f(x, y)\,dx$$

$$y\Big|_{x_n}^{x_{n+1}} = \frac{h}{2}\left\{ f(x_n, y_n) + f(x_{n+1}, y_{n+1}) \right\}$$

$\because$ using Trapezoidal Rule.

$$y_{n+1} - y_n = \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}) \right]$$

$$\Rightarrow y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}) \right]$$

which is improved Euler's Formula. But $f(x_{n+1}, y_{n+1})$ which occurs on R.H.S of equation is unknown. So we first calculate y_{n+1} from Euler's Method which is $y_{n+1} = y_n + h(f(x_n, y_n))$. Thus for each stage we have the following two formulas

Euler's Formula: $y_{n+1}^* = y_n + h[f(x_n, y_n)]$

Improved Euler's Formula

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^*) \right]$$

Example:

Solve $\frac{dy}{dx} = x^2+y^2$; $y(0)=1$, $h=0.1$

Find $y(0.1)$ using Euler's Improved Method

Solution:- $\frac{dy}{dx} = x^2+y^2$

$\Rightarrow f(x,y) = x^2+y^2$

$\Rightarrow f(x_n,y_n) = x_n^2+y_n^2$

$x_0 = 0$, $y_0 = 1$, $h=0.1$

Euler's Formula is

$$y_{n+1}^* = y_n + h\,f(x_n,y_n)$$

$$\Rightarrow y_{n+1}^* = y_n + h\,(x_n^2+y_n^2) \rightarrow ①$$

Euler's Improved Formula is

$$y_{n+1} = y_n + h[f(x_n,y_n) + f(x_{n+1}, y_{n+1}^*)]$$

$$\Rightarrow y_{n+1} = y_n + h[x_n^2 + y_n^2 + x_{n+1}^2 + y_{n+1}^{*2}] \rightarrow ②$$

Put $n=0$ in ①

$$y_1^* = y_0 + h\,(x_0^2+y_0^2)$$
$$= 1 + 0.1\,(0+1)$$
$$y_1^* = 1.1$$

and $x_1 = x_0 + h$
$$= 0 + 0.1 = 0.1$$

Again put $n=0$ in ②

$$y_1 = y_0 + \frac{h}{2}[x_0^2 + y_0^2 + x_1^2 + y_1^{*2}]$$

$$y_1 = 1 + \frac{(0.1)}{2}[0 + 1 + (0.1)^2 + (1.1)^2]$$

$$y_1 = 1 + 0.0500\,(2.2200)$$

$$y_1 = 1.111$$

$$\Rightarrow y(0.1) = 1.111$$

(Ans)

Example:

$\frac{dy}{dx} = x+y$; $y(0)=1$, $h=0.1$

Find $y(0.2)$ using Improved Euler's Method.

Solution:- Given $\frac{dy}{dx} = x+y$

$\therefore f(x,y) = x+y$

$f(x_n,y_n) = x_n+y_n$

$x_0 = 0$, $y_0 = 1$, $h = 0.1$

Euler's Formula:

$$y_{n+1}^* = y_n + h\,f(x_n,y_n)$$

$$y_{n+1}^* = y_n + h\,(x_n+y_n) \rightarrow (i)$$

Improved Euler's Formula:

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n,y_n) + f(x_{n+1},y_{n+1})]$$

$$y_{n+1} = y_n + \frac{h}{2}(x_n + y_n + x_{n+1} + y_{n+1}^*) \rightarrow (ii)$$

I^{st} iteration Put $n=0$ in (i) :

$$y_1^* = y_0 + h(x_0+y_0)$$
$$= 1 + 0.1(0+1) = 1.1$$
$$x_1 = x_0 + h$$
$$x_1 = 0+0.1 = 0.1$$

Put $n=0$ in (ii)

$$y_1 = y_0 + \frac{h}{2}[x_0+y_0+x_1+y_1^*]$$
$$= 1 + \frac{(0.1)}{2}[0+1+0.1+1.1]$$
$$= 1.11 \qquad \because x_1 = 0.1,\ y_1 = 1.11$$

2^{nd} iteration. Put $n=1$ in (i) :

$$y_2^* = y_1 + h(x_1+y_1)$$
$$= 1.11 + 0.1(0.1+1.11)$$
$$y_2^* = 1.231$$

and
$$x_2 = x_0 + 2h$$
$$x_2 = 0+2(0.1) = 0.2$$

Put $n=1$ in (ii)

$$y_2 = y_1 + \frac{h}{2}[x_1+y_1+x_2+y_2^*]$$
$$= 1.11 + \frac{0.1}{2}[0.1+1.11+0.2+1.231]$$
$$y_2 = 1.2421$$
$$x_2 = 0.2$$

$$\Rightarrow \quad y(0.2) = 1.2421$$

(Answer)

II. MODIFIED EULER'S METHOD :-

The modified Euler's formula is

$$y_{n+1} = y_n + h\,f\left(x_n+\frac{h}{2},\ y_n+\frac{h}{2}f(x_n,y_n)\right)$$

Example :

Apply Modified Euler's formula to find $y(0.3)$ given that

$$\frac{dy}{dx} = x+y \;;\; y(0)=1,\ h=0.1$$

Solution :- $\frac{dy}{dx} = x+y \qquad x_0=0,\ y_0=1$

$$f(x,y) = x+y$$
$$f(x_n,y_n) = x_n+y_n$$

Modified Euler's Formula is

$$y_{n+1} = y_n + hf\left[x_n+\frac{h}{2},\ y_n+\frac{h}{2}f(x_n,y_n)\right]$$
$$y_{n+1} = y_n + hf\left[x_n+\frac{h}{2},\ y_n+\frac{h}{2}(x_n+y_n)\right] \rightarrow (*)$$

As $f(x,y) = x+y$

$$f\left(x_n+\frac{h}{2},\ y_n+\frac{h}{2}(x_n+y_n)\right) = x_n+\frac{h}{2}+y_n+\frac{h}{2}(x_n+y_n)$$

Then * becomes

$$y_{n+1} = y_n + h\left(x_n+\frac{h}{2}+y_n+\frac{h}{2}(x_n+y_n)\right)$$
$$= y_n + h\left[\frac{h}{2}[1+x_n+y_n]+x_n+y_n\right]$$

$\Rightarrow y_{n+1} = y_n + h\left[\frac{h}{2}(1+x_n+y_n) + x_n + y_n\right] \rightarrow (*)'$

I^{st} Iteration :- Put $n=0$ in $(*)'$

$y_1 = y_0 + h\left[\frac{h}{2}(1+x_0+y_0) + x_0 + y_0\right]$

$= 1 + 0.1\left[\frac{0.1}{2}(1+0+1) + 0+1\right]$

$= 1 + 0.11$

$= 1.11$

$x_1 = x_0 + h$

$= 0 + 0.1 = 0.1$

II^{nd} Iteration :- Put $n=1$ in $(*)'$

$y_2 = y_1 + h\left[\frac{h}{2}(1+x_1+y_1) + x_1 + y_1\right]$

$= 1.11 + 0.1\left[\frac{0.1}{2}(1+1.11+0.1) + 0.1 + 1.11\right]$

$= 1.11 + 0.1(1.3205)$

$= 1.2421$

$x_2 = x_0 + 2h$

$x_2 = 0 + 2(0.1) = 0.2$

3^{rd} Iteration :- Put $n=2$ in $(*)'$

$y_3 = y_2 + h\left[\frac{h}{2}(1+x_2+y_2) + x_2 + y_2\right]$

$= 1.2421 + 0.1\left[\frac{0.1}{2}(1+0.2+1.2421) + 0.2 + 1.2421\right]$

$y_3 = 1.2421 + 0.1(1.5642)$

$y_3 = 1.3985$

(Answer).

Example :-

Solve $\frac{dy}{dx} = x+2y$; $0 \le x \le 0.2$

$h = 0.1$; $y(0) = 1$

Using Modified Euler's Method.

Solution :- Give $\frac{dy}{dx} = x+2y$

$f(x,y) = x+2y$

$f(x_n, y_n) = x_n + 2y_n$

$x_0 = 0$, $y_0 = 1$, y

we find $y(0.2)$

Euler's Modified Formula is

$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}f(x_n, y_n)\right) \rightarrow (1)$

$f(x_n, y_n) = x_n + 2y_n$

$f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}(x_n+2y_n)\right) = x_n + \frac{h}{2} + 2\left(y_n + \frac{h}{2}(x_n+2y_n)\right)$

Put in (1) we have,

$y_{n+1} = y_n + h\left[x_n + \frac{h}{2} + 2\left(y_n + \frac{h}{2}(x_n+2y_n)\right)\right]$

$y_{n+1} = y_n + h\left[\frac{h}{2} + x_n + 2y_n + h(x_n+2y_n)\right]$

$y_{n+1} = y_n + h\left[\frac{h}{2} + x_n(1+h) + 2y_n(1+h)\right] \longrightarrow (2)$

1^{st} Iteration :- Put n=0 in ②

$$y_1 = y_0 + h\left[\frac{h}{2} + x_0(1+h) + 2y_0(1+h)\right]$$
$$= 1 + 0.1\left[\frac{0.1}{2} + 0(1+0.1) + 2(1+0.1)\right]$$

$$y_1 = 1 + 0.1\left[\frac{0.1}{2} + 2.22\right]$$

$$y_1 = 1.2250$$

And $x_1 = x_0 + h$
$$= 0 + 0.1 = 0.1$$

$x_1 = 0.1$ and $y_1 = 1.2250$
$$y(0.1) = 1.2250.$$

2^{nd} Iteration:- Put $n=1$ in ②

$$y_2 = y_1 + h\left[\frac{h}{2} + x_1(1+h) + 2y_1(1+h)\right]$$

$$y_2 = 1.2250 + 0.1\left[\frac{0.1}{2} + 0.1(1+0.1) + 2(1.2250)(1+0.1)\right]$$

$$= 1.2250 + 0.1[2.8550]$$
$$= 1.2250 + 0.2855$$
$$y_2 = 1.5105$$

$$x_2 = x_0 + 2h$$
$$= 0 + 2(0.1) = 0.2$$

$$\Rightarrow y(0.2) = 1.5105$$

(Answer).

EXERCISE

I Solve following questions by Euler's Method,

(i) $y' = x + y$; $y(0) = 0$, $h = 0.2$
Carry out 6 steps.
Ans: 1.18318

(ii) $\frac{dy}{dx} = x + y^2$ and $y = 1$, at $x = 0$.
Find $y(0.5)$ $\because h = 0.1$
Ans: 1.9346.

(iii) If $xy' = x + y$, when y has at value $x = 1$ is 2.
Find y when $x = 2$ (Step size 0.1)

(iv) Solve $\frac{dy}{dx} = -xy^2$, $y = 2$ at $x = 0$.
obtain y at $x = 0.2$ where $h = 0.05$.

(v) $y' = \frac{-y^2}{1+x^2}$ estimate $y(0.2)$
with $y(0) = 1$, $h = 0.05$.

(vi) Solve $\frac{dy}{dx} = 1 + x\sin(xy)$ $0 \le x \le 1$
with $h = 0.1$ and
$y(0) = 0$.

II Solve following by Improved Euler's Method?

(i) Derive Improved Euler's Method. Apply this to find $y(0.2)$ form $y' = -2xy^2$ with $y(0)=1$, $h=0.1$
Compare your result with exact value

(ii) Find approximate value of y when $x = 0.06$ given $\frac{dy}{dx} = x^2 + y^2$, $y(0)=1$
taking interval 0.02
Ans: 1.0619

(iii) Tabulate $y(x)$ for $x = 0.1, 0.2, 0.3$ from IVP $y' = 1 + xy$ where $y(0)=1$

(iv) Find $y(0.2)$ given
$\frac{dy}{dx} = x + 3y$, $y(0)=1$
$h = 0.1$
Ans: 2.492

III Solve the following question by Modified Euler's Method?

(i) Use Modified Euler's Method to tabulate $y(0.3)$ for $y' = 1 + xy$ where $y(0)=1$

(ii) Solve $\frac{dy}{dx} = x^2 + y^2$, $y(0)=1$, $h=0.5$ at $x=1$ (Hint $y(1)=?$)

(iii) Solve following differential equation
$\frac{dy}{dx} = x - y$ $0 \le x \le 0.2$
$y(0)=1$
taking $h=0.1$ and working to 4 decimal places by modified Euler's Method.
(Ans: 0.8372)
Compare with exact value
Hint: $y(0.2) = ?$
Solution of L.D.E $y' - y = x$
is $y = x - 1 + 2e^{-x}$.

(iv) Show that modified Euler's Method for solution of differential eq.
$\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$
is given by
$$y_{n+1} = y_n + \frac{h}{2}\{f(x_n, y_n) + f(x_n + h, y_n + h f(x_n, y_n))$$
Hint. Use Euler's and improved Euler's Method.

IV Taylor's Series

Algorithm

Consider the differential equation

$$y' = f(x,y) \quad \text{with I.C} \quad y(x_0) = y_0 \quad \rightarrow (i)$$

If $F(x)$ is exact solution of I.V.P given in (i), then Taylor's series of $F(x)$ around $x = x_0$ is given by

$$F(x) = F(x_0 + (x - x_0))$$

$$= F(x_0) + (x - x_0)F'(x_0) + \frac{(x-x_0)^2}{2!}F''(x_0) + \frac{(x-x_0)^3}{3!}F'''(x_0) + \ldots\ldots \rightarrow (2)$$

Put $x = x_1$ and $y_1 = F(x_1)$

(2) $\Rightarrow$

$$y_1 = y_0 + (x_1 - x_0)y_0' + \frac{(x_1-x_0)^2}{2!}y_0'' + \frac{(x_1-x_0)^3}{3!}y_0''' + \frac{(x_1-x_0)^4}{4!}y_0^{(iv)} + \ldots\ldots$$

where $h = x_1 - x_0$.

$$\Rightarrow y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + \ldots\ldots \rightarrow (3)$$

Similarly;

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \frac{h^4}{4!}y_1^{(iv)} + \ldots\ldots$$

so on in general

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2!}y_n'' + \frac{h^3}{3!}y_n''' + \frac{h^4}{4!}y_n^{iv} + \ldots\ldots \rightarrow (*)$$

Remark:

(i) Neglecting h^2 and higher power of h in (*)

$$y_{n+1} = y_n + hy_n'$$

$$y_{n+1} = y_n + hf(x_n, y_n)$$ which is Euler's Formula.

(ii) Neglecting h^3 and higher power of h in (*)

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2!}y_n''$$ which is Taylor's Series Method of order 2.

(iii) Neglecting h^4 and higher power of h in (*)

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2!}y_n'' + \frac{h^3}{3!}y_n'''$$

which is Taylor's Series Method of order 3.

(iv) Put $x_0 = 0$ in (2)

$$F(x) = F(x_0) + (x - x_0)F'(x_0) + \frac{(x-x_0)^2}{2!}F''(x_0) + \ldots\ldots$$

$$y = y_0 + (x - x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \ldots\ldots$$

$$y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \ldots\ldots$$

Which is power series in x of y.

Example :-

Find value of $y(0.3)$ for I.V.P

$y' = -2xy^2$, $y(0) = 1$, $h = 0.1$

using Taylor's Series algorithm of order 2

Solution :-

Taylor's Series algorithm of order 2 is

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2!} y_n'' \longrightarrow (1)$$

$$y' = -2xy^2$$

$$y_n' = -2x_n y_n^2$$

$$y'' = -2x(2yy') + (-2)y^2$$

$$= -4xyy' - 2y^2$$

$$= -4xy(-2xy^2) - 2y^2$$

$$= 8x^2y^3 - 2y^2$$

$$y_n'' = -2y_n^2 + 8x_n^2 y_n^3$$

Put in (1) we have

$$y_{n+1} = y_n + h(-2x_n y_n^2) + \frac{h^2}{2!}(-2y_n^2 + 8x_n^2 y_n^3) \longrightarrow (2)$$

Ist Iteration :- Put $n=0$ in (2)

$$y_1 = y_0 + h(-2x_0 y_0^2) + \frac{h^2}{2!}(-2y_0^2 + 8x_0^2 y_0^3)$$

$$= 1 + (0.1)(-2(0)(1)^2) + \frac{(0.1)^2}{2!}(-2(1)^2 + 8(0)(1)^3)$$

$$y_1 = 1 + 0 + (-0.010) + 0$$

$$y_1 = 0.990, \quad x_1 = x_0 + h = 0 + 0.1 = 0.1$$

IInd Iteration :- Put $n=1$ in (2)

$$y_2 = y_1 + h(-2x_1 y_1^2) + \frac{h^2}{2!}(-2y_1^2 + 8x_1^2 y_1^3)$$

$$y_2 = 0.990 + (0.1)(-2(0.1)(0.99)^2) + \frac{(0.1)^2}{2!}\left[-2(0.99)^2 + 8(0.1)^2(0.99)^3\right]$$

$$y_2 = 0.961$$

$$x_2 = x_0 + 2h$$

$$= 0 + 2(0.1) = 0.2$$

$$\Rightarrow y(0.2) = 0.961.$$

IIIrd Iteration :- Put $n=2$ in (2)

$$y_3 = y_2 + h(-2x_2 y_2^2) + \frac{h^2}{2!}(-2y_2^2 + 8x_2^2 y_2^3)$$

$$y_3 = 0.961 + (0.1)\left[-2(0.2)(0.961)^2\right] + \frac{(0.1)^2}{2!}\left[-2(0.961)^2 + 8(0.2)^2(0.961)^3\right]$$

$$y_3 = 0.961244$$

$$x_3 = x_0 + 3h$$

$$= 0 + 3(0.1) = 0.3$$

$$y(0.3) = 0.961244$$

Exact Value :

$$\frac{dy}{dx} = -2xy^2$$

$$\int \frac{dy}{y^2} = \int -2x\,dx$$

$$\frac{y^{-1}}{-1} = -\frac{2x^2}{2} + c$$

$$-\frac{1}{y} = x^2 + c$$

Given $y(0) = 1$

$$-\frac{1}{y} = 0 + c$$

$$-1 = c \Rightarrow c = -1$$

$$\Rightarrow -\frac{1}{y} = x^2 - 1$$

$$\Rightarrow \frac{1}{y} = 1 + x^2$$

$$y = \frac{1}{1+x^2}$$

$$y(0.3) = \frac{1}{1+(0.3)^2} = 0.917431$$

Error $= 0.917431 - 0.916244$

$= 0.001187$

Ans.

Example :-

If y satisfies eq. $\frac{dy}{dx} = x^2y - 1$, $y(0) = 1$ using Taylor's expension obtain y as series in power of x. Also find $y(0.3)$ to 5 d.p.

Solution:- Taylor's Series

$$y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \frac{x^4}{4!}y_0^{iv} + \cdots \rightarrow (1)$$

when $x_0 = 0$, $y_0 = 1$

Given $y' = x^2y - 1$, $y_0' = x_0^2y_0 - 1$

$= 0(1) - 1 = -1$

$y'' = 2xy + x^2y$, $y_0'' = 2x_0y_0 + x_0^2y_0$

$= 2(0)(1) + (0)(1) = 0$

$y''' = 2y + 2xy' + x^2y'' + 2x'y$

and

$y_0''' = 2y_0 + 2x_0y_0' + x_0^2y_0'' + 2x_0'y_0$

$y_0''' = 2(1) + 2(0)(-1) + (0)^2(0) + 2(0)(1)$

$= 2$

$y'''' = 2y' + 4y' + 4xy'' + 2xy'' + x^2y'''$

$= 6y' + 6xy'' + x^2y'''$

$y_0'''' = 6y_0' + 6x_0y_0'' + x_0^2y_0'''$

$= 6(-1) + 6(0)(2) + (0)(0)$

$= -6$

Put all values in (1).

$$y = 1 + x(-1) + 0 + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \dots\dots$$

$$y = 1 - x + \frac{x^3}{3} - \frac{x^4}{4} + \dots\dots$$

Which is required Taylor's Series expension of y in power of x

Put $x = 0.3$

$$y(0.3) = 1 - (0.3) + \frac{(0.3)^3}{3} - \frac{(0.3)^4}{4} + \dots\dots$$

$$= 1 - 0.3 + 0.0090 - 0.0020$$

$$= 0.7070$$

Answer.

* Example:

Find $y(0.3)$ for I.V.P $y' = x + y$ where $y(0) = 1$. Apply Taylor's Series algorithm of order 3 with $h = 0.1$ Comparing with exact result.

Solution:- As Taylor's Series Algorithm of order 3 is

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' \rightarrow ①$$

Given $y' = x + y \Rightarrow y_n' = x_n + y_n$

$y'' = 1 + y' \Rightarrow y_n'' = 1 + y_n' = 1 + x_n + y_n$

$y''' = 0 + y''$

$y_n''' = 1 + x_n + y_n$

$y_n' = x_n + y_n$

$y_n'' = 1 + x_n + y_n$

$y_n''' = 1 + x_n + y_n$

Put in ①

$$y_{n+1} = y_n + h(x_n + y_n) + \frac{h^2}{2!}(1 + x_n + y_n) + \frac{h^3}{3!}(1 + x_n + y_n) \rightarrow (*)$$

$y(0) = 1$

$x_0 = 0, \quad y_0 = 1, \quad h = 0.1$

1st Iteration: Put $n = 0$ in $(*)$

$$y_1 = y_0 + h(x_0 + y_0) + \frac{h^2}{2!}(1 + x_0 + y_0) + \frac{h^3}{3!}(1 + x_0 + y_0)$$

$$= 1 + (0.1)(0 + 1) + \frac{(0.1)^2}{2!}(1 + 0 + 1) + \frac{(0.1)^3}{3!}(1 + 0 + 1)$$

$y_1 = 1.11033$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

$\Rightarrow y(0.1) = 1.11033$

II Iteration: Put $n = 1$ in $(*)$

$$y_2 = y_1 + h(x_1 + y_1) + \frac{h^2}{2!}(1 + x_1 + y_1) + \frac{h^3}{3!}(1 + x_1 + y_1)$$

$$= 1.11033 + (0.1)(0.1 + 1.1103) + \frac{(0.1)^2}{2!}(1 + 0.1 + 1.1103)$$

$$+ \frac{(0.1)^3}{3!}(1 + 0.1 + 1.11033)$$

$y_2 = 1.2427$

$x_2 = x_0 + 2(0.1) = 0.2$

$y(0.2) = 1.2427$

III^{rd} Iteration:- Put $n = 2$ in (*)

$y_3 = y_2 + h(x_2 + y_2) + \frac{h^2}{2!}(1 + x_2 + y_2) + \frac{h^3}{3!}(1 + x_2 + y_2)$

$y_3 = 1.2427 + 0.1(0.2 + 1.2427) +$
$\frac{(0.1)^2}{2!}(1 + 0.2 + 1.2427) +$
$\frac{(0.1)^3}{3!}(1 + 0.2 + 1.2427)$

$y_3 = 1.3996$

$x_3 = x_0 + 3h$
$= 0 + 3(0.1)$
$= 0.3$

$y(0.3) = 1.399679$

Ans.

Exact Solution:

$$\frac{dy}{dx} = x + y$$

$$\frac{dy}{dx} - y = x$$

I.F $e^{-\int dx} = e^{-x}$

$$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}x$$

$$\int \frac{d}{dx}(e^{-x}y) = \int e^{-x}x$$

$$e^{-x}y = -xe^{-x} - e^{-x} + c$$

$$y = -x + Ce^{x} - 1$$

using condition

$y(0) = 1$

$1 = -(0) + Ce^{0} - 1$

$2 = C$

$\Rightarrow y = -x - 1 + 2e^{x}$

put $x = 0.3$

$y = -0.3 - 1 + 2e^{(0.3)}$

$y = 1.399718$

Error = Exact − Appro. value

$= 1.39971 - 1.3996$

$= 0.000039$

(Answer)

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Exercise

(i) Using Taylor's Series algorithm of order 2 find $y(0.2)$, $y(0.4)$ and $y(0.6)$ of I.V.P is given that

$$y' = -xy^2 , \quad y(0) = 2.$$

Compare with exact value.

(ii) Find approximate value of $y(0.1)$, $y(0.2)$ of $\frac{dy}{dx} = (x+1)y$; $y(0) = 1$.

using T.S algorithm of order 3 $(h = 0.1)$

(iii) Using Taylor's Series find solution of

$$y' = -2xy^2 \quad y(0) = 1$$

at $x = 0.3$ correct to 5 d.p.

(iv) Using T.S find solution of

$$xy' = x - y \quad ; \quad y(2) = 2$$

at $x = 2.1$ correct to 5 d.p

(Hint: $y' = 1 - \frac{y}{x}$)

(v) Given $y' = y^2 + 1$; $y(0) = 0$ obtain y as a Series in power of x. Also find $y(0.2)$ check your answer with exact solution.

Ans: $y(0.2) = 0.202709$

and $y = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\ldots$

V Runge - Kutta Method

(R.K Method)

Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \quad ; \quad y(x_0) = y_0$$

Integrate from x_n to x_{n+1}

$$\int_{x_n}^{x_{n+1}} \frac{dy}{dx} = \int_{x_n}^{x_{n+1}} f(x, y)\,dx$$

$$y\Big|_{x_n}^{x_{n+1}} = \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

$\because$ by Trapezoidal Rule

$$y(x_{n+1}) - y(x_n) = \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

$$y_{n+1} - y_n = \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

$$= y_n + \frac{h}{2}\left[f(x_n, y_n) + f\left(x_n + h, y_n + hf(x_n, y_n)\right)\right] \rightarrow (i)$$

$\because$ using Euler's Formula

Let $k_1 = hf(x_n, y_n)$

$k_2 = hf[x_n + h, y_n + hf(x_n, y_n)]$

or $k_2 = hf(x_n + h, y_n + k_1)$

put in (i)

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

which is R. K Formula of order 2.

Runge-Kutta Method of order 2

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

where $k_1 = hf(x_n, y_n)$

$k_2 = hf(x_n + h, y_n + k_1)$

Runge-Kutta Method of Order 4

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$k_1 = hf(x_n, y_n)$

$k_2 = hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$

$k_3 = hf(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$

$k_4 = hf(x_n + h, y_n + k_3)$

Runge-Kutta Method of Order 3

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

where

$k_1 = hf(x_n, y_n)$

$k_2 = hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$

$k_3 = hf(x_n + h, y_n - k_1 + 2k_2)$

Example: Apply Runge formula of order 2 to approximate value of y when $x = 1.2$ given $\frac{dy}{dx} = 3x + y^2$ and $y = 1.2$ when $x = 1$

Solution:- $\frac{dy}{dx} = 3x + y^2$

$f(x, y) = 3x + y^2$

$f(x_n, y_n) = 3x_n + y_n^2$

R.K method of order 2 is

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

where $k_1 = hf(x_n, y_n)$

$k_2 = hf(x_n + h, y_n + k_1)$

Ist iteration:- $x_0 = 1$, $y_0 = 1.2$, $h = 0.1$

$k_1 = hf(x_0, y_0) = h(3x_0 + y_0^2) = 0.1(3(1) + (1.2)^2)$

$k_1 = 0.444$

$k_2 = hf(x_n + h, y_n + k_1)$

$= hf(1 + 0.1, 1.2 + 0.444)$

$= hf(1.1, 1.644)$

$= 0.1\{3(1.1) + (1.644)^2\}$

$= 0.600$

Runge's Formula is

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2) \rightarrow (i)$$

Put $n = 0$ in (i)

$$y_1 = y_0 + \frac{1}{2}(0.444 + 0.600)$$

$$y_1 = 1.722$$

$x = x_0 + h$

$= 1 + 0.1 = 1.1$

$\Rightarrow y(1.1) = 1.722$

2nd iteration:-

$x_1 = 1.1, \quad y_1 = 1.722, \quad h = 0.1$

$k_1 = h f(x_1, y_1)$

$= h(3x_1 + y_1^2)$

$= 0.1[3(1.1) + (1.722)^2]$

$k_1 = 0.6265$

and

$k_2 = h f(x_1 + h, y_1 + k_1)$

$= 0.1 f(1.1 + 0.1, 1.722 + 0.6265)$

$= 0.1 f(1.2, 2.3485)$

$= 0.1\{3(1.2) + (2.3485)^2\}$

$= 0.1[3.6 + 5.5155]$

$= 0.91155$

Put $n = 1$ in (i)

$y_2 = y_1 + \frac{1}{2}(k_1 + k_2)$

$y_2 = 1.722 + \frac{1}{2}(0.6265 + 0.91155)$

$= 1.722 + 0.769$

$y_2 = 2.491$

$x_2 = x_0 + 2h$

$= 1 + 2(0.1) = 1.2$

$x_2 = 1.2$

$y(1.2) = 2.491$

(Ans)

Example:

Solve following differential equation using R.K method of order 4

$$\frac{dy}{dx} = 1 + y^2 \ ; \quad y(0) = 0$$

$$h = 0.2$$

Find $y(0.2)$ and $y(0.4)$. Compare with exact value.

Solution:- Given $\frac{dy}{dx} = 1 + y^2$

$f(x, y) = 1 + y^2$

$f(x_n, y_n) = 1 + y_n^2$

$x_0 = 0, \quad y_0 = 0, \quad h = 0.2.$

Step I: By R.K method of order 4

$k_1 = h f(x_0, y_0) = h(1 + y_0^2) = 0.2(1 + 0^2)$

$k_1 = 0.2$

$k_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$= (0.2) f(0 + 0.2/2, 0 + 0.2/2)$

$= 0.2 f(0.1, 0.1)$

$= 0.2(1 + (0.1)^2)$

$= 0.20204$

$k_3 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$

$= h f(0 + \frac{0.2}{2}, 0 + \frac{0.2020}{2})$

$= 0.2 f(0.1, 0.101)$

$= 0.2(1 + (0.101)^2)$

$= 0.2020402$

$K_4 = hf(x_0+h, y_0+k_3)$
$= 0.2 f(0+0.2, 0+0.2020402)$
$= 0.2\{1+(0.2020402)^2\}$
$= 0.208164048$

$y_1 = y_0 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)$
$= 0 + \frac{1}{6}[0.2 + 2(0.2020) + 2(0.2020402) + 0.20816408]$
$y_1 = 0.202707$
$x_1 = x_0 + h$
$= 0 + 0.2 = 0.2$
$\Rightarrow y(0.2) = 0.202707$ (Ans).

Step II :-

$x_1 = 0.2,\ y_1 = 0.202707,\ h = 0.2$
$K_1 = hf(x_1, y_1)$
$= 0.2 f(0.2, 0.202707)$
$= 0.2\{1+(0.202707)^2\}$
$= 0.2082181$

$K_2 = hf(x_1+\frac{h}{2}, y_1+\frac{k_1}{2})$
$= hf(0.2+\frac{0.2}{2}, 0.202707+\frac{0.2082181}{2})$
$= hf(0.3, 0.306816)$
$= 0.2\{1+(0.306816)^2\}$
$= 0.2182272$

$K_3 = hf(x_1+\frac{h}{2}, y_1+\frac{k_2}{2})$
$= 0.2 f[0.2+\frac{0.2}{2}, 0.202707+\frac{0.2182272}{2})$
$= 0.2 f(0.3, 0.3121206)$
$= 0.2\{1+(0.3121206)^2\}$
$= 0.2194839$

$K_4 = hf(x_1+h, y_1+k_3)$
$= 0.2 f(0.2+0.2, 0.202707+0.2194839)$
$= 0.2 f(0.4, 0.4221909)$
$= 0.2[1+(0.4221909)^2]$
$= 0.235649$

$y_2 = y_1 + \frac{1}{6}[k_1+2k_2+2k_3+k_4]$
$= 0.202707 + \frac{1}{6}[0.2082181 + 2(0.2182272) + 2(0.2194839) + 0.235649]$

$y_2 = 0.422791$
$x_2 = x_0 + 2h$
$= 0 + 2(0.2) = 0.4$
$y(0.4) = 0.422791$

Exact Value :

$\frac{dy}{dx} = 1+y^2$

$\frac{dy}{1+y^2} = dx$

$\int \frac{dy}{1+y^2} = \int dx$

$\tan^{-1}(y) = x + c$

$y(0) = 0$

$\tan^{-1}(0) = 0 + c$

$\Rightarrow c = 0$

$\Rightarrow \tan^{-1} y = x$

$y = \tan x$

$y(0.2) = \tan(0.2)$

$= 0.2027$

and $y(0.4) = \tan(0.4)$

$= 0.4227932$

(Answer)

Example :-

Show that most popular R.K method of order 4 reduces to Simpson's Rule in $\frac{dy}{dx}$ is a function of x alone.

Solution :- $\frac{dy}{dx} = f(x)$

$\therefore \quad K_1 = hf(x_n) = hf_n$

$K_2 = hf(x_n + \frac{h}{2}) = hf_{n+\frac{h}{2}}$

$K_3 = hf(x_n + \frac{h}{2}) = hf_{n+\frac{h}{2}}$

$K_4 = hf(x_n + h) = hf_{n+h}$

Thus

$y_{n+1} = y_n + \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$

$= y_n + \frac{1}{6}\left[hf_n + 2hf_{n+\frac{h}{2}} + 2hf_{n+\frac{h}{2}} + hf_{n+h}\right]$

$y_{n+1} = y_n + \frac{h}{6}\left(f_n + 4f_{n+\frac{h}{2}} + f_{n+h}\right)$

which is Simpson's Rule with step size $\frac{h}{2}$.

Replace h by $2h$.

$y_{n+1} = y_n + \frac{h}{3}\left(f_n + 4f_{n+h} + f_{n+2h}\right)$

$y_{n+1} = y_n + \frac{h}{3}\left\{f(x_n) + 4f(x_n + h) + f(x_n + 2h)\right\}$

which is Simpson's $\frac{1}{3}$ Rule.

(proved)

Example :-

Use Runge-kutta 4^{th} order method to find an approximate value of y for $x = 0.2$ in step of 0.1 if $\frac{dy}{dx} = x + y^2$ given that $y(0) = 1$.

Solution :- $\frac{dy}{dx} = x + y^2$, $y(0) = 1$

$f(x, y) = x + y^2$

$x_0 = 0$, $y_0 = 1$, $h = 0.1$

Step I :

$K_1 = hf(x_0, y_0) = h(x_0 + y_0^2)$

$= 0.1(0 + 1^2)$

$= 0.100$

$K_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2})$

$= hf(0 + \frac{0.1}{2}, \frac{0.100}{2} + 1)$

$= hf(0.05, 1.05)$

$= 0.1(0.05 + (1.05)^2)$

$= 0.1152$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$

$= 0.1 f(0 + \frac{0.1}{2}, \frac{0.1152}{2} + 1)$

$= 0.1 f(0.05, 1.0576)$

$= 0.1[0.05 + (1.0576)^2]$

$= 0.1168$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= 0.1 f(0 + 0.1, 1 + 0.1168)$

$= 0.1[0.1 + (1.1168)^2]$

$= 0.1347$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1 + \frac{1}{6}[0.100 + 2(0.1152) + 2(0.1168) + 0.1347)]$

$= 1.1165$

$x_1 = x_0 + h$

$= 0 + 0.1 = 0.1$

$y(0.1) = 1.1165$

Step II: $x_1 = 0.1$, $y_1 = 1.1165$, $h = 0.1$

$k_1 = hf(x_1, y_1) = 0.1 f(0.1, 1.1165)$

$= 0.1[0.1 + (1.1165)^2]$

$= 0.1347$

$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2})$

$= 0.1 f[0.1 + \frac{0.1}{2}, 1.1165 + \frac{0.1347}{2}]$

$= 0.1[f(0.15, 1.1838)]$

$= 0.1[0.15 + (1.1838)^2]$

$= 0.1551$

$k_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2})$

$= hf(0.1 + \frac{0.1}{2}, 1.1165 + \frac{0.1551}{2})$

$= 0.1 f(0.15, 1.194)$

$= 0.1[0.15 + (1.194)^2]$

$= 0.1576$

$k_4 = hf(x_1 + h, y_1 + k_3)$

$= 0.1 f(0.1 + 0.1, 1.1165 + 0.1576)$

$= 0.1 f(0.2, 1.2741)$

$= 0.1[0.2 + (1.2741)^2]$

$= 0.1823$

$y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 1.1165 + \frac{1}{6}[0.1347 + 2(0.1551) + 2(0.1576) + (0.1823)]$

$= 1.2736$

$x_2 = x_0 + 2h$

$= 0 + 2(0.1) = 0.2$

$\Rightarrow y(0.2) = 1.2736$

(Answer).

EXERCISE

(i) Derive R-K method of order 2 and use it to find $y(0.02)$ and $y(0.05)$ with $h = 0.01$ of problem
$y' = 2y + x$, $y(0) = 1$

(ii) Apply Runge's Formula (2^{nd} order) to find an approximate value of y when $x = 1.1$ given that
$\frac{dy}{dx} = x - y$ & $y = 1$ when $x = 1$

Ans: 1.005

(iii) Calculate $y(0.1)$ and $y(0.2)$ from I.V.P
$y' = x + 2y$, $y(0) = 0.75$
taking $h = 0.1$
Comparing with exact value.

Ans. $y(0.1) = 0.92$
$y(0.2) = 1.1384$

(iv) Solve equation $y' = x + y$, $y(0) = 0$ at $x = 0.5$ Taking $h = 0.1$ compare your result with exact value

(v) Apply Rung's formula (2^{nd} order) to find value of y when $x = 0.02$ given that $\frac{dy}{dx} = x^2 + y$ & $y(0) = 1$

Ans: 1.0202

(vi) Solve I.V.P by Runge's Formula of order 4 of following

(i) Obtain y when $x = 1.1$ given that $y = 1.2$ when $x = 1$ and y satisfies equation
$\frac{dy}{dx} = 3x + y^2$

Ans: 1.7271

(ii) Find $y(0.2)$ for eq. $\frac{dy}{dx} = \frac{y - x}{y + x}$, $y(0) = 1$
Take $h = 0.2$

Ans: 1.167845

(iii) At $x = 0.8$ find approx. value of y given that $y = 0.41$ when $x = 0.4$ and
$\frac{dy}{dx} = \sqrt{x + y}$

Ans: 0.8489

(iv) Find (0.2) given
$\frac{dy}{dx} = 3x + \frac{1}{2}y$, $y(0) = 1$, $h = 0.1$

Ans: 1.17492

(v) Use R.K Method find y when $x = 1.2$ in step size 0.1 given that

(i) $\frac{dy}{dx} = \frac{y}{x}$ and $y(1) = 1$

Ans: 1.2

(ii) $\frac{dy}{dx} = x^2 + y^2$, $y(1) = 1.5$

Ans: ?

Predictor-Corrector Methods

The methods discussed so far to solve differential equations numerically were self starting one step method. To apply these methods, we were required information only at the begining of interval. But, now in present section we shall discuss Predictor-corrector methods which require function value at point $x_n, x_{n-1}, x_{n-2}, \ldots$ for the computation of function at x_{n+1}. And these function value can be obtained by using Euler's method, Taylor's method, R.K method of order 4 etc.

A predictor formula is used to predict the value of y at x_{n+1} and then a corrector formula is used to improve the value of y. Here we discuss two methods as following

(i) Milne's Method
(ii) Adam-Bashforth Method.

(i) Milne's Method :-

To solve IVP

$$\frac{dy}{dx} = f(x, y) \quad \text{where } y(x_0) = y_0$$

h = Step size

by Milne's method, we first find the approximate value of y_{n+1} by predictor formula and then improve it by corrector formula.

In terms of f, Milne's Predictor and corrector formula are

Predictor formula :

$$\tilde{y}_{n+1} = y_{n-3} + \frac{4h}{3}\left[2f_{n-2} - f_{n-1} + 2f_n\right] \longrightarrow (A)$$

Corrector formula :

$$y_{n+1} = y_{n-1} + \frac{h}{3}\left[f_{n-1} + 4f_n + \tilde{f}_{n+1}\right]$$

where h is step size.

and $\tilde{f}_{n+1} = f(x_{n+1}, \tilde{y}_{n+1})$

(Also $\tilde{y}_{n+1} = y^P_{n+1}$)
P stands for predictor.

Example.

Find $y(0.4)$ and $y(0.5)$ by Milne's method from

$$\frac{dy}{dx} = xy \quad ; \quad y(0) = 1$$
$$h = 0.1$$

Solution :- The given IVP is

$$\frac{dy}{dx} = xy \quad ; \quad y(0) = 1$$
$$h = 0.1$$

By R.K method of order 4 is

$$y_{n+1} = y_n + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right] \rightarrow (i)$$

Where $k_1 = hf(x_n, y_n)$

$k_2 = hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$

$k_3 = hf(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$

$k_4 = hf(x_n + h, y_n + k_3)$

For $n=0$,

$k_1 = hf(x_0, y_0) = h(x_0 y_0) = (0.1)[(0)(1)] = 0$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$= h\left(x_0 + \frac{h}{2}\right)\left(y_0 + \frac{k_1}{2}\right)$

$= (0.1)(0+0.5)(1+0) = 0.005$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$

$= h(x_0 + h/2)(y_0 + k_2/2)$

$= (0.1)(0+0.5)(1+0.0025) = 0.0050125$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= h(x_0 + h)(y_0 + k_3)$

$= (0.1)(0+0.1)(1+0.0050125)$

$= 0.1005013$

So by (i) we have

$y_1 = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$

$= 1 + \frac{1}{6}\left[0 + 2(0.005) + 2(0.0050125) + 0.1005013\right]$

$y(0.1) = 1.00501252$

For $n=1$:-

$k_1 = hf(x_1, y_1)$

$= h x_1 y_1 = (0.1)(0.1)(1.005013)$

$= 0.010050$

$k_2 = hf(x_1 + h/2, y_1 + k_1/2)$

$= h\left(x_1 + \frac{h}{2}\right)\left(y_1 + \frac{k_1}{2}\right)$

$= (0.1)(0.1+0.05)\left(1.005013 + \frac{0.01005}{2}\right)$

$= (0.1)(0.15)(1.010038) = 0.015151$

$k_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2})$

$= h(x_1 + h/2)(y_1 + k_2/2)$

$= (0.1)(0.1+0.05)\left(1.005013 + \frac{0.015151}{2}\right)$

$= 0.015189$

$$k_4 = hf(x_1+h,\ y_1+k_3)$$
$$= h\,(x_1+h)(y_1+k_3)$$
$$= (0.1)(0.1+0.1)(1.005013+0.015189)$$
$$= 0.020404$$

So (i) gives

$$y_2 = y_1 + \frac{1}{6}\left[k_1+2k_2+2k_3+k_4\right]$$

$$= 1.005013 + \frac{1}{6}\Big[0.01005 + 2(0.01515+0.01519) + 0.020404\Big]$$

$$= 1.005013 + \frac{1}{6}(0.091134)$$

$$= 1.005013 + 0.015189$$

$$y(0.2) = 1.020202$$

For $n=2$:-

$$k_1 = hf(x_2, y_2)$$
$$= h\,(x_2)(y_2) = (0.1)(0.2)(1.020202)$$
$$= 0.020404$$

$$k_2 = hf\left(x_2+\frac{h}{2},\ y_2+\frac{k_1}{2}\right)$$

$$= (0.1)(0.2+0.5)\left(1.020202+\frac{0.020404}{2}\right)$$
$$= 0.025760$$

$$k_3 = hf\left(x_2+\frac{h}{2},\ y_2+\frac{k_2}{2}\right)$$

$$= h\left(x_2+\frac{h}{2}\right)\left(y_2+\frac{k_2}{2}\right)$$

$$= (0.1)(0.2+0.5)\left(1.020202+\frac{0.025760}{2}\right)$$

$$= 0.025827$$

$$k_4 = hf(x_2+h,\ y_2+k_3)$$
$$= h\,(x_2+h)(y_2+k_3)$$
$$= (0.1)(0.2+0.1)(1.020202+0.025827)$$
$$= 0.031338$$

So (i) gives

$$y_3 = y_2 + \frac{1}{6}\left[k_1+2(k_2+k_3)+k_4\right]$$

$$= 1.020202 + \frac{1}{6}\Big[0.020404 + 2(0.02576+0.025827) + 0.031381\Big]$$

$$= 1.020202 + \frac{1}{6}(0.154959)$$

$$= 1.020202 + 0.025827$$

$$y(0.3) = 1.046029$$

For $n=3$:-

$$k_1 = hf(x_3, y_3) = h\,(x_3)(y_3)$$
$$= (0.1)(0.3)(1.046029)$$
$$= 0.031381$$

$$k_2 = hf\left(x_3+\frac{h}{2},\ y_3+\frac{k_1}{2}\right)$$

$$= h\,(x_3+h/2)(y_3+k_1/2)$$
$$= (0.1)(0.3+0.5)\left(1.040629+\frac{0.031381}{2}\right)$$
$$= 0.037160$$

$$k_3 = hf\left(x_3+\frac{h}{2},\ y_3+\frac{k_2}{2}\right)$$
$$= h\left(x_3+\frac{h}{2}\right)\left(y_3+\frac{k_2}{2}\right)$$

$$= (0.1)(0.3+0.5)\left(1.040629+\frac{0.031760}{2}\right)$$

$$= 0.037261$$

$$k_4 = hf(x_3+h,\ y_3+k_3)$$

$= h(x_3+h)(y_3+k_3)$
$= (0.1)(0.3+0.1)(1.046029+0.037261)$
$= 0.043332$

So (i) gives

$$y_4 = y_3 + \frac{1}{6}[k_1 + 2(k_2+k_3)+k_4]$$
$$= 1.046029 + \frac{1}{6}[0.03138 + 2(0.037162 + 0.037261) + 0.043332]$$
$$= 1.046029 + \frac{1}{6}(.224558)$$
$$y(0.4) = 1.083288$$

For $n=4$:-

$k_1 = hf(x_4, y_4) = h(x_4)(y_4)$
$= (0.1)(0.4)(1.083288)$
$= 0.043332$

$k_2 = hf(x_4+\frac{h}{2}, y_4+\frac{k_1}{2})$
$= h(x_4+\frac{h}{2})(y_4+\frac{k_1}{2})$
$= (0.1)(0.4+0.5)(1.083288+\frac{0.04332}{2})$
$= 0.49723$

$k_3 = hf(x_4+\frac{h}{2}, y_4+\frac{k_2}{2})$
$= (0.1)(0.4+0.5)(1.083288+\frac{0.49723}{2})$
$= 0.049867$

$k_4 = hf(x_4+h, y_4+k_3)$
$= h(x_4+h)(y_n+k_3)$
$= (0.1)(0.4+0.1)(1.083288+0.049867)$
$= 0.056658$

So (i) gives

$$y_5 = y_4 + \frac{1}{6}[k_1 + 2(k_2+k_3)+k_4]$$
$$= 1.083288 + \frac{1}{6}[0.043332 + 2(0.049723 + 0.049867) + 0.056658]$$
$$= 1.083288 + \frac{1}{6}(0.299170)$$
$$\Rightarrow y(0.5) = 1.133150.$$

Now

$f(x_n, y_n) = x_n y_n$
$f(x_1, y_1) = x_1 y_1 = (0.1)(1.005013)$
$= 0.100501$
$f(x_2, y_2) = x_2 y_2 = (0.2)(1.020202)$
$= 0.20404$
$f_3 = f(x_3, y_3) = x_3 y_3 = (0.3)(1.046029)$
$= 0.313809$
$f_4 = x_4 y_4 = (0.4)(1.083288)$
$= 0.433315$
$f_5 = x_5 y_5 = (0.5)(1.133150)$
$= 0.566575$

So by Milne's Predictor and Corrector formula, we have

For $n=4$: By Predictor formula, (A)

$$\tilde{y}_4 = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$$
$$= 1 + \frac{4(0.1)}{3}[2(0.100501) - 0.20404) + 2(0.313809)]$$
$$\tilde{y}_4 = 1.083277.$$
$$\tilde{f}_4 = x_4 \tilde{y}_4 = (0.4)(1.083277)$$
$$= 0.433311$$

and by Corrector formula by (B)

$$\tilde{y}_4 = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + \tilde{f}_4\right]$$

$$= 1.020202 + \frac{0.1}{3}\left[0.20404 + 4(0.313809) + 0.433311\right]$$

$$= 1.020202 + \frac{0.1}{3}(1.892587)$$

$$y(0.4) = 1.083288$$

For $n=4$:

By Predictor formula, (A)

$$\tilde{y}_5 = y_1 + \frac{4h}{3}\left[2f_2 - f_3 + 2f_4\right]$$

$$= 1.005013 + \frac{4(0.1)}{3}\left[2(0.20404) - 0.313809 + 2(0.433315)\right]$$

$$= 1.005013 + \frac{0.4}{3}(0.960901)$$

$$= 1.133133$$

$$\Rightarrow \tilde{f}_5 = x_5 \tilde{y}_5 = (0.5)(1.133133) = 0.566567$$

By Corrector formula, (B)

$$y_5 = y_3 + \frac{h}{3}\left[f_3 + 4f_4 + \tilde{f}_5\right]$$

$$= 1.046029 + \frac{0.1}{3}\left[0.313809 + 4(0.433315) + 0.566567\right]$$

$$= 1.046029 + \frac{0.1}{3}\left[2.613636\right]$$

$$y(0.5) = 1.13315$$

(Answer)

Example :-

Find $y(0.4)$ by Milne's method from $\frac{dy}{dx} = x+y$, $y(0)=1$, $h=0.1$

Solution :-

The given IVP is

$$\frac{dy}{dx} = x+y \; ; \quad y(0)=1, \; h=0.1$$

Then by Taylor's Method of order 4 is

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \frac{h^4}{4!} y_n^{iv} \rightarrow (1)$$

For $n=0$:

$$y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv}$$

given $y' = x+y \Rightarrow y_0' = x_0 + y_0 = 0+1 = 1$

$y'' = 1+y' \Rightarrow y_0'' = 1+y_0' = 1+1 = 2$

And by Corrector formula by (B)

$y_4 = y_2 + \frac{h}{3}[f_2 + 4f_3 + \tilde{f_4}]$

$= 1.020202 + \frac{0.1}{3}[0.20404 + 4(0.313809) + 0.433311]$

$= 1.020202 + \frac{0.1}{3}(1.892587)$

$y(0.4) = 1.083288$

For $n=4$:

By Predictor formula, (A)

$\tilde{y_5} = y_1 + \frac{4h}{3}[2f_2 - f_3 + 2f_4]$

$= 1.005013 + \frac{4(0.1)}{3}[2(0.20404) - 0.313809 + 2(0.433315)]$

$= 1.005013 + \frac{0.4}{3}(0.960901)$

$= 1.133133$

$\Rightarrow \tilde{f_5} = x_5 \tilde{y_5}$

$= (0.5)(1.133133)$

$= 0.566567$

By Corrector formula, (B)

$y_5 = y_3 + \frac{h}{3}[f_3 + 4f_4 + \tilde{f_5}]$

$= 1.046029 + \frac{0.1}{3}[0.313809 + 4(0.433311) + 0.566567]$

$= 1.046029 + \frac{0.1}{3}[2.613636]$

$y(0.5) = 1.13315$

(Answer)

Example :-

Find $y(0.4)$ by Milne's method from $\frac{dy}{dx} = x + y$, $y(0) = 1$, $h = 0.1$

Solution :-

The given IVP is

$\frac{dy}{dx} = x + y$; $y(0) = 1$, $h = 0.1$

Then by Taylor's Method of order 4 is

$y_{n+1} = y_n + h y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \frac{h^4}{4!} y_n^{iv} \rightarrow (1)$

For $n=0$:

$y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv}$

Given $y' = x + y \Rightarrow y_0' = x_0 + y_0 = 0 + 1 = 1$

$y'' = 1 + y' \Rightarrow y_0'' = 1 + y_0'$

$= 1 + 1$

$= 2$

$y''' = y''$

$= y_0'' = 2$

$y^{(iv)} = y_0^{(iv)} = y'''_0 = 2$

So by (i) gives

$$y_1 = 1 + (0.1)(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(2) + \frac{(0.1)^4}{4!}(2)$$

$= 1 + 0.1 + 0.01 + 0.000333 + 0.000008$

$= 1.110341$

$y(0.1) = 1.110341$

For $n = 1$:

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \frac{h^4}{4!}y_1^{(iv)} \longrightarrow \text{(ii)}$$

$y_1' = x_1 + y_1$

$= 0.1 + 1.110341 = 1.210341$

$y_1'' = 1 + y_1'$

$= 1 + 1.210341 = 2.210341$

$y_1''' = y_1'' = 2.210341$

$y_1^{(iv)} = y_1''' = 2.21034$

So (ii) gives

$$y_2 = 1.110341 + (0.1)(1.210341) + \frac{(0.1)^2}{2!}(2.21034)$$

$$+ \frac{(0.1)^3}{3!}(2.21034) + \frac{(0.1)^4}{4!}(2.21034)$$

$y(0.2) = 1.242804$

For $n = 2$:

$$y_3 = y_2 + hy_2' + \frac{h^2}{2!}y_2'' + \frac{h^3}{3!}y_2''' + \frac{h^4}{4!}y_2^{(iv)} \longrightarrow \text{(iii)}$$

$y_2' = x_2 + y_2 = 0.2 + 1.242804$

$= 1.442804$

$y_2'' = 1 + y_2' = 1 + 1.442804 = 2.442804$

$y_2''' = y_2'' = 2.442804$

$y_2^{(iv)} = y_2''' = 2.442804$

So (iii) gives

$$y_3 = 1.242804 + (0.1)(1.442804) + \frac{(0.1)^2}{2!}(2.442804)$$

$$+ \frac{(0.1)^3}{3!}(2.442804) + \frac{(0.1)^4}{4!}(2.442804)$$

$= 1.242804 + 0.14428 + 0.012214 +$

$0.000407 + 0.000010$

$y(0.3) = 1.399715$

For $n = 4$:-

$$y_4 = y_3 + hy_3' + \frac{h^2}{2!}y_3'' + \frac{h^3}{3!}y_3''' + \frac{h^4}{4!}y_3^{(iv)} \longrightarrow \text{(iv)}$$

$y_3' = x_3 + y_3 = 0.3 + 1.399715 = 1.699715$

$y_3'' = 1 + y_3' = 1 + 1.399715 = 2.399715$

$y_3''' = y_3'' = 2.399715$

$y_3^{(iv)} = y_3''' = 2.399715$.

So (iv) gives

$$y_4 = 1.399715 + (0.1)(1.699715) + \frac{(0.1)^2}{2!}(2.399715)$$
$$+ \frac{(0.1)^3}{3!}(2.399715) + \frac{(0.1)^4}{4!}(2.399715)$$

$$y(0.4) = 1.399715 + 0.169972 + 0.011999$$
$$+ 0.000400 + 0.000010$$
$$y(0.4) = 1.582096.$$

Now by Milne's Predictor and Corrector formula we have

For $n = 3$:

By Predictor formula

$$\tilde{y}_4 = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

Now

$$f(x_1, y_1) = f_1 = x_1 + y_1 = 0.1 + 1.110341$$
$$= 1.210341$$

$$f_2 = x_2 + y_2 = 0.2 + 1.242804$$
$$= 1.442804$$

$$f_3 = x_3 + y_3 = 0.3 + 1.399715$$
$$= 1.699715$$

So

$$\tilde{y}_4 = 1 + \frac{4(0.1)}{3}\left[2(1.210341) - 1.442804 + 2(1.699715)\right]$$

$$= 1 + \frac{0.4}{3}\left[4.377308\right]$$

$$\tilde{y}_4 = 1.583641$$

Now $\tilde{f}_4 = x_4 + \tilde{y}_4$
$$= 0.4 + 1.583641 = 1.983641$$

By Corrector formula

$$y_4 = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + \tilde{f}_4\right]$$

$$= 1.242804 + \frac{0.1}{3}\left[1.442804 + 4(1.699715) + 1.98364)\right]$$

$$= 1.242804 + \frac{0.1}{3}(10.225305)$$

$$\Rightarrow y(0.4) = 1.583648.$$

(Answer).

(ii) ADAM-BASHFORTH METHOD :-

Predictor formula :

$$y^p_{n+1} = y_n + \frac{h}{24}\left[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}\right] \rightarrow (i)$$

Corrector formula :

$$y_{n+1} = y_n + \frac{h}{24}\left[9f^p_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}\right] \rightarrow (ii)$$

Where $f^p_{n+1} = f(x_{n+1}, y^p_{n+1})$

* Example:

Solve $xy' = x - y$ to find $y(2.2)$ and $y(2.25)$
given $y(2) = 2$, $y(2.05) = 2.00061$,
$y(2.1) = 2.002385$, $y(2.15) = 2.005242$

Solution: $xy' = x - y$

$$y' = \frac{x - y}{x}$$

$$\Rightarrow y' = 1 - \frac{y}{x}$$

$$f(x, y) = 1 - \frac{y}{x}$$

$x_0 = 2$, $x_1 = 2.05$, $x_2 = 2.1$
$y_0 = 2$, $y_1 = 2.00061$, $y_2 = 2.002385$

$x_3 = 2.15$
$y_3 = 2.005242$

$f_0 = f(x_0, y_0) = f(2, 2) = 1 - \frac{2}{2} = 1 - 1 = 0$

$f_1 = f(x_1, y_1) = f(2.05, 2.00061)$
$= 1 - \frac{2.00061}{2.05} = 0.0240927$

$f_2 = f(x_2, y_2) = f(2.1, 2.002385)$
$= 1 - \frac{2.002385}{2.1}$
$= 0.0464834$

$f_3 = f(x_3, y_3) = f(2.15, 2.005242)$
$= 1 - \frac{2.005242}{2.15}$
$= 0.0673294$

Adam Bashforth Predictor formula

$$y^p_{n+1} = y_n + \frac{h}{24}\left[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}\right] \rightarrow (i)$$

and Corrector formula is

$$y_{n+1} = y_n + \frac{h}{24}\left[9f^p_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}\right] \rightarrow (ii)$$

Put $n=3$ in (i)

$$y_4^p = y_3 + \frac{h}{24}[55f_3 - 59f_2 + 37f_1 - 9f_0]$$

$$y_4^p = 2.005242 + \frac{0.05}{24}[55(0.0673294) - 59(0.0464834) + 37(0.0240927) - 9(0)]$$

$$\Rightarrow y_4^p = 2.0091003$$

$$f_4^p = f(x_4, y_4^p) = f(2.2, 2.0091003) = 1 - \frac{2.0091003}{2.2} = 0.0867726$$

Again put $n=3$ in Corrector formula (ii)

$$y_4 = y_3 + \frac{h}{24}[9f_4^p + 19f_3 - 5f_2 + f_1]$$

$$y_4 = 2.00524 + \frac{0.05}{24}[9(0.867726) + 19(0.673294) - 5(0.0464834) + 0.0240927]$$

$$y_4 = 2.005242 + 0.003858$$

$$y_4 = 2.0091$$

$$\Rightarrow y(2.2) = 2.0091 \quad \text{(Answer)}$$

Put $n=4$ in Predictor formula (i)

$$y_5^p = y_4 + \frac{h}{24}[55f_4 - 59f_3 + 37f_2 - 9f_1]$$

$$y_5^p = 2.00913 + \frac{0.05}{24}[55(0.0867728) - 59(0.0673294) + 37(0.0464834) - 9(0.0240927)]$$

$$y_5^p = 2.01392$$

$$f_5^p = f(x_5, y_5^p) = f(2.25, 2.01392) = 1 - \frac{2.01392}{2.25} = 0.1049245$$

Again put $n=4$ in Corrector formula (ii)

$$y_5 = y_4 + \frac{h}{24}[9f_5^p + 19f_4 - 5f_3 + f_2]$$

$$y_5 = 2.0091 + \frac{0.05}{24}[9(0.1049245) + 19(0.867728) - 5(0.0673294) + 0.0464834]$$

$$y_5 = 2.01389$$

$$\Rightarrow y(2.25) = 2.01389$$

(Answer)

"Exercise"

Apply Milne's and Adam- Bashforth predictor corrector formulae on following questions.

(i) Solve $y' = -xy^2$ at $x = 0.8$ and $x = 1.0$

using starting values
$y(0) = 2$, $y(0.2) = 1.92308$
$y(0.4) = 1.72414$, $y(0.6) = 1.47059$.

Ans: $y(0.8) = 1.21808$,
$y(1.0) = 1.0$

(ii) $2\frac{dy}{dx} = (1+x^2)y^2$ and $y(0) = 1$
$y(0.1) = 1.06$
$y(0.2) = 1.12$
$y(0.3) = 1.21$

Evaluate $y(0.4)$

Ans: 1.2797

(iii) Solve $y' = y^2$ where $y(1) = 1$ and $h = 0.1$

find value of IVP at $x = 1.5$

Ans: $y(1.5) = 2$

System of differential equations.

A Set of Simultaneous equations that involves two or more unknown function and their derivatives is called system of differential equations.

For example:

$$\frac{dx}{dt} = 2x + y + t$$

$$\frac{dy}{dt} = x - y^2$$

Here t is independent variable and x, y are dependent variable.

Runge - kutta Method of order 4 :-

To solve equations

$$\frac{dx}{dt} = f(t, x, y)$$

$$\frac{dy}{dt} = g(t, x, y)$$

$x(t_0) = x_0$, $y(t_0) = y_0$

we find k_1, k_2, k_3, k_4 and l_1, l_2, l_3, l_4 in the order

$k_1 = h f(t_n, x_n, y_n)$

$l_1 = h g(t_n, x_n, y_n)$

$k_2 = h f(t_n + h/2, x_n + k_1/2, y_n + l_1/2)$

$l_2 = h g(t_n + \frac{h}{2}, x_n + \frac{k_1}{2}, y_n + \frac{l_1}{2})$

$k_3 = h f(t_n + \frac{h}{2}, x_n + \frac{k_2}{2}, y_n + \frac{l_2}{2})$

$l_3 = h g(t_n + \frac{h}{2}, x_n + \frac{k_2}{2}, y_n + \frac{l_2}{2})$

$k_4 = h f(t_n + h, x_n + k_3, y_n + l_3)$

$l_4 = h g(t_n + h, x_n + k_3, y_n + l_3)$

Finally we obtain

$$x_{n+1} = x_n + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$y_{n+1} = y_n + \frac{1}{6}\left[l_1 + 2l_2 + 2l_3 + l_4\right]$$

Runge-kutta Method of Order 2:-

$k_1 = h f(t_n, x_n, y_n)$

$l_1 = h g(t_n, x_n, y_n)$

$k_2 = h f(t_n + h, x_n + k_1, y_n + l_1)$

$l_2 = h g(t_n + h, x_n + k_1, y_n + l_1)$

$$x_{n+1} = x_n + \frac{1}{2}(k_1 + k_2)$$

$$y_{n+1} = y_n + \frac{1}{2}(l_1 + l_2).$$

Example :- Solve initial value problem at $t = 1.2$ where

$$\frac{dy}{dt} = x - 2y, \quad \frac{dx}{dt} = 2x + y.$$

$x(1) = 2, \quad y(1) = 3.$

Solution :- $f(t, x, y) = 2x + y$

$g(t, x, y) = x - 2y$

$\therefore f(t_n, x_n, y_n) = 2x_n + y_n$

$g(t_n, x_n, y_n) = x_n - 2y_n$

I^{st} Iteration :-

$t_0 = 1, \ x_0 = 2, \ y_0 = 3$

Let $h = 0.1$

$k_1 = h f(t_0, x_0, y_0)$

$= 0.1 \ (2x_0 + y_0)$

$= 0.1 \{2(2) + 3\}$

$= 0.7$

$l_1 = h g(t_0, x_0, y_0)$

$= h \ (x_0 - 2y_0)$

$= 0.1 \{2 - 2(3)\}$

$= -0.4$

$k_2 = h f(t_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}, y_0 + \frac{l_1}{2})$

$= h \{2(x_0 + \frac{k_1}{2}) + y_0 + \frac{l_1}{2}\}$

$= (0.1)\{2 \times 2.35 + 2.8\} = 0.75$

$$l_2 = hg\left(t_0+\frac{h}{2},\ x_0+\frac{k_1}{2},\ y_0+\frac{l_1}{2}\right)$$

$$= 0.1\,g(1.05,\ 2.35,\ 2.8)$$
$$= 0.1\,\{2.35 - 2(2.8)\}$$
$$= -0.325$$

$$k_3 = hf\left(t_0+\frac{h}{2},\ x_0+\frac{k_2}{2},\ y_0+\frac{l_2}{2}\right)$$

$$= 0.1\,f(1.05,\ 2.375,\ 2.8375)$$
$$= 0.1\,\{2\times 2.375 + 2.8375\}$$
$$= 0.75875$$

$$l_3 = hg\left(t_0+\frac{h}{2},\ x_0+\frac{k_2}{2},\ y_0+\frac{l_2}{2}\right)$$

$$= 0.1\,g(1.05,\ 2.375,\ 2.8375)$$
$$= 0.1\,\{2.375 - 2(2.8375)\}$$
$$= -0.33$$

$$k_4 = hf(t_0+h,\ x_0+k_3,\ y_0+l_3)$$
$$= 0.1\,f(1.1,\ 2.75875,\ 2.67)$$
$$= 0.1\,\{2(2.7587) + 2.67\}$$
$$= 0.81875$$

$$l_4 = hg(t_0+h,\ x_0+k_3,\ y_0+l_3)$$
$$= 0.1\,g(1.1,\ 2.75875,\ 2.67)$$
$$= 0.1\,\{2.75875 - 2(2.67)\}$$
$$= -0.25812$$

$$x_1 = x_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$x_1 = 2 + \frac{1}{6}\{0.7 + 2(0.75 + 0.75875) + 0.81875\}$$
$$= 2.75604$$

$$y_1 = y_0 + \frac{1}{6}[l_1 + 2(l_2 + l_3) + l_4]$$

$$= 3 + \frac{1}{6}[-0.4 + 2(-0.325 - 0.33) - 0.258125]$$
$$= 2.67198$$

2^{nd} Iteration :-

$$t_1 = t_0 + h = 1 + 0.1 = 1.1$$
$$x_1 = 2.75604$$
$$y_1 = 2.67198$$

$$k_1 = hf(t_1, x_1, y_1)$$
$$l_1 = hg(t_1, x_1, y_1)$$

So on Same Solve as in I^{st} iteration replace x_0, y_0 by x_1, y_1 and find k_1, k_2, k_3, k_4 and l_1, l_2, l_3, l_4 we get

$$x_2 = 3.650467$$
$$y_2 = 2.47814.$$

(Answer).

* Example :

Solve the equations

$$\frac{dy}{dx} = xz + 1 \quad , \quad \frac{dz}{dx} = -xy$$

for $x = 0.3, 0.3, 0.9$

Given $y = 0$, $z = 1$, when $x = 0$.

Solution.— $x = 0.3, 0.3, 0.9$ means $h = 0.3$

I^{st} find value at 0.3 then at 0.6 and 0.9.

Note that x is independent and y, z are dependent variables.

$$f(x,y,z) = xz + 1 \quad , \quad g(x,y,z) = -xy$$

$$f(x_n, y_n, z_n) = x_n z_n + 1 \quad , \quad g(x_n, y_n, z_n) = -x_n y_n$$

I^{st} iteration :–

$x_0 = 0$

$y_0 = 0$

$z_0 = 1$ $\qquad h = 0.3$

$$k_1 = hf(x_0, y_0, z_0)$$
$$= h[x_0 z_0 + 1]$$
$$= h[(0)(1) + 1]$$
$$= 0.3[0 + 1] = 0.3$$

$$l_1 = hg(x_0, y_0, z_0)$$
$$= h[-x_0 y_0]$$
$$= 0.3[-(0)(0)]$$
$$= 0$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$
$$= 0.3 f(0.15, 0.15, 1)$$
$$= 0.3\{0.15 \times 1 + 1\} = 0.3450$$

$$l_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$
$$= 0.3\, g(0.15, 0.15, 1)$$
$$= 0.3[-0.15 \times 0.15]$$
$$= -0.00675$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$
$$= 0.3 f(0.15, 0.1725, 0.9966)$$
$$= 0.3[0.15 \times 0.9966 + 1]$$
$$= 0.3448$$

$$l_3 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$
$$= 0.3\, g(0.15, 0.1725, 0.9966)$$
$$= 0.3[-0.15 \times 0.1725]$$
$$= -0.00776$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$
$$= 0.3 f(0.3, 0.3448, 0.99224)$$
$$= 0.3[0.3 \times 0.9924 + 1]$$
$$= 0.3893$$

$$l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3)$$
$$= 0.3\, g(0.3, 0.3448, 0.99224)$$

$= 0.3 \{-0.3 \times 0.3448\}$

$= -0.031036$

$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 0 + \frac{1}{6}[0.3 + 2(0.3450 + 0.3448) + 0.3893]$

$= 0.3448$

$z_1 = z_0 + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$

$= 1 + \frac{1}{6}[0 + 2(-0.00675 + 0.3893) + (-0.031036)]$

$= 1 - 0.00100l$

$= 0.9899$

$\Rightarrow y(0.3) = 0.3448$

IInd Iteration :-

$x_1 = x_0 + h = 0 + 0.3 = 0.3$

$y_1 = 0.3448, \; z_1 = 0.9899$

By using R.k method order 4 formula we have

$k_1 = 0.3891, \quad l_1 = -0.031042$

$k_2 = 0.43155, \quad l_2 = -0.07281$

$k_3 = 0.42873, \quad l_3 = 0.07568$

$k_4 = 0.4646, \quad l_4 = -0.1393$

$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 0.3448 + \frac{1}{6}(0.3891 + 2(0.43155 + 0.4287) + 0.4646)$

$= 0.3448 + 0.4290$

$y_2 = 0.7738$

$z_2 = z_1 + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$

$= 0.9899 + (-0.07788)$

$= 0.9121$

3rd Iteration :-

$x_2 = x_1 + h = 0.3 + 0.3 = 0.6$

$y_2 = 0.7738$

$z_2 = 0.9121$

$k_1 = 0.4642, \quad l_1 = -0.1393$

$k_2 = 0.4896, \quad l_2 = -0.2263$

$k_3 = 0.47976, \quad l_3 = -0.2292$

$k_4 = 0.4844, \quad l_4 = -0.3386$

$y_3 = y_2 + \frac{1}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$

$= 0.7738 + 0.4812 = 1.2550$

$z_3 = z_2 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

$= 0.9121 - 0.23148 = 0.6806$

$\Rightarrow y(0.9) = 1.2550$ (Answer)

Exercise

Solve the following system of equations. Find x and y two steps after the given value of t.

(a) $\dfrac{dx}{dt} = x + y - t$

$\dfrac{dy}{dt} = 3x - y + 2t$ $\qquad x(0) = 1,\ y(0) = 2$

Ans: $x_1 = 1.31722,\ y_1 = 2.15000$

$x_2 = 1.67828,\ y_2 = 2.40134$

where $h = 0.1$

(b) $\dfrac{dx}{dt} = x^2 + y,\quad \dfrac{dy}{dt} = 2x - 3y + t$

$x(0) = 0.2,\ y(0) = 0.1$

Ans: $x_1 = 0.21495,\ y_1 = 0.11450$

$, h = 0.1$

(c) $\dfrac{dx}{dt} = 0.1x + 0.2y - 0.3t$

$\dfrac{dy}{dt} = 0.2x - 0.1y + 0.2t$

at $t = 1.1$ where $x(1) = 1,\ y(1) = 2.$

GOOD LUCK